

The Canonical Basis of $\dot{\mathbf{U}}$ for Type A_2

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Abstract

The modified quantized enveloping algebra has a remarkable basis, called the canonical basis, which was introduced by Lusztig. In this paper, all these monomial elements of the canonical basis for type A_2 are determined and we also give a conjecture about all polynomial elements of the canonical basis.

Keywords: the modified quantized enveloping algebra ; canonical basis ; the quasi-R-matrix.

Canonical base of quantized enveloping algebras was introduced by Lusztig and have many remarkable properties. However it is hard to compute the base. By now, the base was only computed for type A_2 , A_3 , B_2 , see [L1, X2, X1]. For type A_4 , part of the base was computed in [HYY, HY, LH].

A modified form of the quantized enveloping algebras was introduced by Lusztig and a remarkable basis, called the canonical basis, was also given in [L2, L3]. However it is even harder to compute the basis. By now, the base was only computed for type A_1 in Lusztig's book [L3], which are consisting of two monomial elements.

In the present paper, we determine all these monomial elements of the canonical basis for type A_2 and also give a conjecture about all polynomial elements of the canonical basis. We hope that the results of the paper will be helpful to understand the canonical basis of the modified quantized enveloping algebra.

1 Preliminaries

1.1

We will need some notations. Let a be an integer and b an positive integer. Set

$$[a] = \frac{v^a - v^{-a}}{v - v^{-1}}, [b] = \prod_{h=1}^b \frac{v^h - v^{-h}}{v - v^{-1}}, \begin{bmatrix} a \\ b \end{bmatrix} = \prod_{h=1}^b \frac{v^{a-h+1} - v^{-(a-h+1)}}{v^h - v^{-h}}.$$

We have

$$\begin{bmatrix} a+b \\ b \end{bmatrix} = \frac{[a+b]!}{[a]![b]!} \quad \text{for } a, b \in \mathbb{N}.$$

Let Y, X be the coroot lattice and the weight lattice of the root datum $(I, <, >, Y, X)$ corresponding to the quantized enveloping algebra $U_v(sl_3)$. By definition, $\mathbf{U} = U_v(sl_3)$ is an associative algebra over $\mathbb{Q}(v)$ (v an indeterminate) with the generators $e_i, f_i, k_\mu; 1 \leq i \leq 2, \mu \in Y$, subject to the following relations

$$k_0 = 1, k_{\mu_1} k_{\mu_2} = k_{\mu_1 + \mu_2}, \quad \text{for all } \mu_1, \mu_2 \in Y; \quad (1)$$

$$k_\mu e_i k_\mu^{-1} = v^{<\mu, \alpha_i>} e_i, k_\mu f_i k_\mu^{-1} = v^{-<\mu, \alpha_i>} f_i, \quad \text{for all } i = 1, 2, \mu \in Y; \quad (2)$$

$$e_i f_j - f_j e_i = \delta_{i,j} \frac{k_i - k_i^{-1}}{v - v^{-1}} \quad (3)$$

$$\sum_{r+s=2} (-1)^r e_i^{(r)} e_j e_i^{(s)} = 0, \quad \text{for } i \neq j; \quad (4)$$

$$\sum_{r+s=2} (-1)^r f_i^{(r)} f_j f_i^{(s)} = 0, \quad \text{for } i \neq j; \quad (5)$$

where $e_i^{(r)}$ denotes the divided power $\frac{e_i^r}{[r]!}$, and $k_i = k_{\alpha_i}$.

There is a unique isomorphism of $\mathbb{Q}(v)$ -vector spaces $\sigma : \mathbf{U} \rightarrow \mathbf{U}$ such that $\sigma(e_i) = e_i, \sigma(f_i) = f_i, \sigma(k_\mu) = k_\mu^{-1}$ for $\mu \in Y$ and $\sigma(uu') = \sigma(u')\sigma(u)$ for $u, u' \in \mathbf{U}$.

We define a category \mathcal{C} as follows. An object of \mathcal{C} is a \mathbf{U} -module M with a given direct sum decomposition $M = \bigoplus_{\lambda \in X} M_\lambda$ (as a $\mathbb{Q}(v)$ -vector space) such that, for any $\mu \in Y, \lambda \in X$ and $m \in M_\lambda$, we have $k_\mu m = v^{<\mu, \lambda>} m$. The subspaces M_λ are called the weight spaces of M . A morphism in \mathcal{C} is a \mathbf{U} -linear map.

If $M', M'' \in \mathcal{C}$, the tensor product $M' \otimes M''$ is naturally a $\mathbf{U} \otimes \mathbf{U}$ -module with $(u' \otimes u'')(m' \otimes m'') = u' m' \otimes u'' m''$. We restrict it to a \mathbf{U} -module via the algebra homomorphism $\Delta : \mathbf{U} \rightarrow \mathbf{U} \otimes \mathbf{U}$ (Δ is the comultiplication). The resulting \mathbf{U} -module is naturally an object of \mathcal{C} . If $m' \in M'_{\lambda'}, m'' \in M''_{\lambda''}$, we have the following identities:

$$e_i^{(a)}(m' \otimes m'') = \sum_{a'+a''=a} v^{a'a''+a''<i, \lambda'>} e_i^{(a')} m' \otimes e_i^{(a'')} m'';$$

$$f_i^{(a)}(m' \otimes m'') = \sum_{a'+a''=a} v^{a'a''-a'<i, \lambda''>} f_i^{(a')} m' \otimes f_i^{(a'')} m''.$$

Let $\mathbf{U}^+, \mathbf{U}^-$ be the positive part and negative part of \mathbf{U} respectively. It is known that if \mathbf{f} is the associative $\mathbb{Q}(v)$ -algebra generated by $\theta_i, 1 \leq i \leq 2$ which satisfy the analogue of relation for \mathbf{U}

$$\sum_{r+s=2} (-1)^r \theta_i^r \theta_j \theta_i^s = 0, \quad \text{for } i \neq j.$$

Then the natural morphism $\mathbf{f} \rightarrow \mathbf{U}^\pm, \theta_i \rightarrow e_i, \theta_i \rightarrow f_i$ are all algebra isomorphisms.

1.2

Next let us recall the definition of the modified quantized enveloping algebra $\dot{\mathbf{U}}$, which is the modified form of \mathbf{U} .

If $\lambda', \lambda'' \in X$, we set

$${}_{\lambda'}\mathbf{U}_{\lambda''} = \mathbf{U} / \left(\sum_{\mu \in Y} (k_\mu - v^{<\mu, \lambda'>}) \mathbf{U} + \sum_{\mu \in Y} \mathbf{U} (k_\mu - v^{<\mu, \lambda'>}) \right).$$

Let $\pi_{\lambda', \lambda''} : \mathbf{U} \rightarrow {}_{\lambda'}\mathbf{U}_{\lambda''}$ be the canonical projection. Then we define

$$\dot{\mathbf{U}} = \bigoplus_{\lambda', \lambda'' \in X} {}_{\lambda'}\mathbf{U}_{\lambda''}.$$

There is a natural associative $\mathbb{Q}(v)$ -algebra structure on $\dot{\mathbf{U}}$ inherited from that of \mathbf{U} . The elements $1_\lambda = \pi_{\lambda, \lambda}(1)$ ($\lambda \in X$) of $\dot{\mathbf{U}}$ satisfy

$$1_\lambda 1_{\lambda'} = \delta_{\lambda, \lambda'} 1_\lambda.$$

Then we have

$${}_{\lambda'}\mathbf{U}_{\lambda''} = 1_{\lambda'} \dot{\mathbf{U}} 1_{\lambda''}.$$

The algebra $\dot{\mathbf{U}}$ does not generally 1, but instead a collection of orthogonal idempotents.

We have the following identities in $\dot{\mathbf{U}}$

$$e_i^{(a)} 1_\lambda = 1_{\lambda + a\alpha_i} e_i^{(a)}, f_i^{(b)} 1_\lambda = 1_{\lambda - b\alpha_i} f_i^{(b)} \text{ for } 1 \leq i \leq 2, \lambda \in X, a, b \geq 0.$$

If ω_1, ω_2 is the fundamental weight in X , then any element of X can be written as $\lambda = \lambda_1 \omega_1 + \lambda_2 \omega_2, \lambda_1, \lambda_2 \in \mathbb{Z}$. We will denote this element by $\lambda = (\lambda_1, \lambda_2)$. So the identities above can be written as

$$\begin{aligned} e_1^{(a)} 1_{(\lambda_1, \lambda_2)} &= 1_{(\lambda_1 + 2a, \lambda_2 - a)} e_1^{(a)}, \quad e_2^{(a)} 1_{(\lambda_1, \lambda_2)} = 1_{(\lambda_1 - a, \lambda_2 + 2a)} e_2^{(a)}. \\ f_1^{(b)} 1_{(\lambda_1, \lambda_2)} &= 1_{(\lambda_1 - 2b, \lambda_2 + b)} f_1^{(b)}, \quad f_2^{(b)} 1_{(\lambda_1, \lambda_2)} = 1_{(\lambda_1 + b, \lambda_2 - 2b)} f_2^{(b)}. \end{aligned}$$

The map $\sigma : \mathbf{U} \rightarrow \mathbf{U}$ induces, for each λ', λ'' , a linear isomorphism ${}_{\lambda'}\mathbf{U}_{\lambda''} \rightarrow {}_{-\lambda''}\mathbf{U}_{-\lambda'}$. Taking direct sums, we obtain a linear isomorphism $\sigma : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$ such that $\sigma(1_\lambda) = 1_{-\lambda}$ for all $\lambda \in X$, and $\sigma(uxu') = \sigma(u')\sigma(x)\sigma(u)$ for all $u, u' \in \mathbf{U}$ and $x \in \dot{\mathbf{U}}$.

Let us denote $\dot{\mathbf{B}}$ the canonical basis of $\dot{\mathbf{U}}$ following [L3]. We have the following theorem, which was conjectured by Lusztig in [L4] and had been proved by Kashiwara in [K, theorem 4.3.2].

Theorem 1.2. for any element $b \in \dot{\mathbf{B}}$, we have $\sigma(b) \in \dot{\mathbf{B}}$.

1.3

Assume $V(a\omega_1 + b\omega_2)$ is the finite dimensional irreducible highest weight \mathbf{U} -module of the highest weight vector $\eta_{(a,b)}$, $a, b \geq 0$, ω_1, ω_2 is the fundamental weight as above; $V(-s\omega_1 - t\omega_2)$ is the finite dimensional irreducible lowest weight \mathbf{U} -module of the lowest weight vector $\xi_{(-s,-t)}$, $s, t \geq 0$. Then

$$\dim V(a\omega_1 + b\omega_2) = \frac{(a+1)(b+1)(a+b+2)}{2}.$$

As is well known, the canonical basis of \mathbf{U}^+ is given in [L1] by Lusztig as follows

$$e_2^{(u)} e_1^{(v)} e_2^{(w)}, e_1^{(u)} e_2^{(v)} e_1^{(w)}, v \geq u + w, u, v, w \in \mathbb{N}.$$

where $e_2^{(u)} e_1^{(u+w)} e_2^{(w)} = e_1^{(w)} e_2^{(u+w)} e_1^{(u)}$ is taken only once.

The canonical basis of the \mathbf{U}^- has a remarkable property that it also gives the canonical basis $\mathbf{B}(a\omega_1 + b\omega_2)$ of the highest weight module $V(a\omega_1 + b\omega_2)$ of \mathbf{U} via the action on the highest weight vector; Of course, the canonical basis of the \mathbf{U}^+ gives the canonical basis $\mathbf{B}(-s\omega_1 - t\omega_2)$ of the lowest weight module $V(-s\omega_1 - t\omega_2)$ of \mathbf{U} via the action on the lowest weight vector. By [L3, Theorem 14.4.11], we can get that $\mathbf{B}(a\omega_1 + b\omega_2)$ consists of the following elements:

$$f_2^{(u)} f_1^{(v)} f_2^{(w)} \eta_{(a,b)}, \text{ where } 0 \leq w \leq b, 0 \leq u \leq a, u + w \leq v \leq a + w.$$

$$f_1^{(s)} f_2^{(t)} f_1^{(r)} \eta_{(a,b)}, \text{ where } 0 \leq s \leq b - 1, 0 \leq r \leq a, s + 1 + r \leq t \leq b + r.$$

of course, if $b = 0$, we only take the first part.

2 Monomial elements

The main result of this note is the following theorem on all monomial elements of the canonical basis $\dot{\mathbf{B}}$ of $\dot{\mathbf{U}}$.

Theorem 2. All the monomials of the canonical basis $\dot{\mathbf{B}}$ are given by the following list

Part (1)

$$e_1^{(k)} 1_{(l,m)} f_2^{(u)} f_1^{(v)} f_2^{(w)}, -l \geq v + k - u, v \geq u + w. \quad (1)$$

$$f_2^{(u)} f_1^{(v)} f_2^{(w)} 1_{(l,m)} e_1^{(k)}, -l \leq w - k - v, v \geq u + w. \quad (2)$$

$$e_2^{(h)} e_1^{(k)} 1_{(l,m)} f_2^{(u)} f_1^{(v)} f_2^{(w)}, -l \geq v + k - u, -m \geq u + h - k, h \leq k, v \geq u + w. \quad (3)$$

$$f_2^{(u)} f_1^{(v)} f_2^{(w)} 1_{(l,m)} e_2^{(h)} e_1^{(k)}, -l \leq w + h - k - v, -m \leq -w - h, h \leq k, v \geq u + w. \quad (4)$$

$$e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_2^{(u)} f_1^{(v)} f_2^{(w)}, -l \geq v - u + k - j, -m \geq u + j, j \leq k, v \geq u + w. \quad (5)$$

$$f_2^{(u)} f_1^{(v)} f_2^{(w)} 1_{(l,m)} e_1^{(k)} e_2^{(j)}, -l \leq w - k - v, -m \leq -w + k - j, j \leq k, v \geq u + w. \quad (6)$$

$$e_2^{(h)} e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_2^{(u)} f_1^{(v)} f_2^{(w)}, -l \geq v+k-j-u, -m \geq u+j, k \geq h+j, v \geq u+w. \quad (7)$$

$$f_2^{(u)} f_1^{(v)} f_2^{(w)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} e_2^{(j)}, -l \leq w-v+h-k, -m \leq -w-h, k \geq h+j, v \geq u+w. \quad (8)$$

$$e_1^{(k)} 1_{(l,m)} f_1^{(u)} f_2^{(v)} f_1^{(w)}, -l \geq u+k, v \geq u+w. \quad (9)$$

$$f_1^{(u)} f_2^{(v)} f_1^{(w)} 1_{(l,m)} e_1^{(k)}, -l \leq -k-w, v \geq u+w. \quad (10)$$

$$e_2^{(h)} e_1^{(k)} 1_{(l,m)} f_1^{(u)} f_2^{(v)} f_1^{(w)}, -l \geq u+k, -m \geq v-u+h-k, h \leq k, v \geq u+w. \quad (11)$$

$$f_1^{(u)} f_2^{(v)} f_1^{(w)} 1_{(l,m)} e_2^{(h)} e_1^{(k)}, -l \leq -w+h-k, -m \leq w-v-h, h \leq k, v \geq u+w. \quad (12)$$

$$e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_1^{(u)} f_2^{(v)} f_1^{(w)}, -l \geq u+k-j, -m \geq v-u+j, j \leq k, v \geq u+w. \quad (13)$$

$$f_1^{(u)} f_2^{(v)} f_1^{(w)} 1_{(l,m)} e_1^{(k)} e_2^{(j)}, -l \leq -w-k, -m \leq w-v+k-j, j \leq k, v \geq u+w. \quad (14)$$

$$e_2^{(h)} e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_1^{(u)} f_2^{(v)} f_1^{(w)}, -l \geq u+k-j, -m \geq j+v-u, k \geq h+j, v \geq u+w. \quad (15)$$

$$f_1^{(u)} f_2^{(v)} f_1^{(w)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} e_2^{(j)}, -l \leq h-k-w, -m \leq w-v-h, k \geq h+j, v \geq u+w. \quad (16)$$

$$e_2^{(h)} e_1^{(k)} 1_{(l,m)} f_1^{(v)} f_2^{(w)}, -l \geq v+k, -m \geq w-v+h-k, h \leq k, v \geq w. \quad (17)$$

$$f_2^{(u)} f_1^{(v)} 1_{(l,m)} e_1^{(k)} e_2^{(j)}, -l \leq -k-v, -m \leq v-u+k-j, j \leq k, v \geq u. \quad (18)$$

$$e_2^{(h)} e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_1^{(v)}, -l \geq v+k-j, k \geq h+j, v \geq 0. \quad (19)$$

$$f_1^{(v)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} e_2^{(j)}, -l \leq -v+h-k, k \geq h+j, v \geq 0. \quad (20)$$

$$e_2^{(h)} e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_1^{(v)} f_2^{(w)}, -l \geq v+k-j, -m \geq w-v+j, k \geq h+j, v \geq w. \quad (21)$$

$$f_2^{(u)} f_1^{(v)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} e_2^{(j)}, -l \leq -v+h-k, -m \leq v-u-h, k \geq h+j, v \geq u. \quad (22)$$

$$e_1^{(k)} 1_{(l,m)} f_2^{(v)}, k \geq 0, v \geq 0. \quad (23)$$

$$e_1^{(k)} 1_{(l,m)} f_2^{(v)} f_1^{(w)}, -l \geq w-v+k, k \geq 0, v \geq w. \quad (24)$$

$$f_1^{(u)} f_2^{(v)} 1_{(l,m)} e_1^{(k)}, -l \leq v-u-k, k \geq 0, v \geq u. \quad (25)$$

$$e_2^{(h)} e_1^{(k)} 1_{(l,m)} f_2^{(v)}, -m \geq v+h-k, h \leq k, v \geq 0. \quad (26)$$

$$f_2^{(v)} 1_{(l,m)} e_1^{(k)} e_2^{(j)}, -m \leq k-j-v, j \leq k, v \geq 0. \quad (27)$$

$$e_2^{(h)} e_1^{(k)} 1_{(l,m)} f_2^{(v)} f_1^{(w)}, -l \geq w - v + k, -m \geq v + h - k, h \leq k, v \geq w. \quad (28)$$

$$f_1^{(u)} f_2^{(v)} 1_{(l,m)} e_1^{(k)} e_2^{(j)}, -l \leq v - u - k, -m \leq -v + k - j, j \leq k, v \geq u. \quad (29)$$

$$e_2^{(h)} e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_2^{(v)}, -m \geq j + v, k \geq h + j, v \geq 0. \quad (30)$$

$$f_2^{(v)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} e_2^{(j)}, -m \leq -v - h, k \geq h + j, v \geq 0. \quad (31)$$

$$e_2^{(h)} e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_2^{(v)} f_1^{(w)}, -l \geq w - v + k - j, -m \geq j + v, k \geq h + j, v \geq w. \quad (32)$$

$$f_1^{(u)} f_2^{(v)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} e_2^{(j)}, -l \leq v - u + h - k, -m \leq -v - h, k \geq h + j, v \geq u. \quad (33)$$

Part (2)

$$e_2^{(k)} 1_{(m,l)} f_1^{(u)} f_2^{(v)} f_1^{(w)}, -l \geq v + k - u, v \geq u + w. \quad (34)$$

$$f_1^{(u)} f_2^{(v)} f_1^{(w)} 1_{(m,l)} e_2^{(k)}, -l \leq w - k - v, v \geq u + w. \quad (35)$$

$$e_1^{(h)} e_2^{(k)} 1_{(m,l)} f_1^{(u)} f_2^{(v)} f_1^{(w)}, -l \geq v + k - u, -m \geq u + h - k, h \leq k, v \geq u + w. \quad (36)$$

$$f_1^{(u)} f_2^{(v)} f_1^{(w)} 1_{(m,l)} e_1^{(h)} e_2^{(k)}, -l \leq w + h - k - v, -m \leq -w - h, h \leq k, v \geq u + w. \quad (37)$$

$$e_2^{(k)} e_1^{(j)} 1_{(m,l)} f_1^{(u)} f_2^{(v)} f_1^{(w)}, -l \geq v - u + k - j, -m \geq u + j, j \leq k, v \geq u + w. \quad (38)$$

$$f_1^{(u)} f_2^{(v)} f_1^{(w)} 1_{(m,l)} e_2^{(k)} e_1^{(j)}, -l \leq w - k - v, -m \leq -w + k - j, j \leq k, v \geq u + w. \quad (39)$$

$$e_1^{(h)} e_2^{(k)} e_1^{(j)} 1_{(m,l)} f_1^{(u)} f_2^{(v)} f_1^{(w)}, -l \geq v + k - j - u, -m \geq u + j, k \geq h + j, v \geq u + w. \quad (40)$$

$$f_1^{(u)} f_2^{(v)} f_1^{(w)} 1_{(m,l)} e_1^{(h)} e_2^{(k)} e_1^{(j)}, -l \leq w - v + h - k, -m \leq -w - h, k \geq h + j, v \geq u + w. \quad (41)$$

$$e_2^{(k)} 1_{(m,l)} f_2^{(u)} f_1^{(v)} f_2^{(w)}, -l \geq u + k, v \geq u + w. \quad (42)$$

$$f_2^{(u)} f_1^{(v)} f_2^{(w)} 1_{(m,l)} e_2^{(k)}, -l \leq -k - w, v \geq u + w. \quad (43)$$

$$e_1^{(h)} e_2^{(k)} 1_{(m,l)} f_2^{(u)} f_1^{(v)} f_2^{(w)}, -l \geq u + k, -m \geq v - u + h - k, h \leq k, v \geq u + w. \quad (44)$$

$$f_2^{(u)} f_1^{(v)} f_2^{(w)} 1_{(m,l)} e_1^{(h)} e_2^{(k)}, -l \leq -w + h - k, -m \leq w - v - h, h \leq k, v \geq u + w. \quad (45)$$

$$e_2^{(k)} e_1^{(j)} 1_{(m,l)} f_2^{(u)} f_1^{(v)} f_2^{(w)}, -l \geq u+k-j, -m \geq v-u+j, j \leq k, v \geq u+w. \quad (46)$$

$$f_2^{(u)} f_1^{(v)} f_2^{(w)} 1_{(m,l)} e_2^{(k)} e_1^{(j)}, -l \leq -w-k, -m \leq w-v+k-j, j \leq k, v \geq u+w. \quad (47)$$

$$e_1^{(h)} e_2^{(k)} e_1^{(j)} 1_{(m,l)} f_2^{(u)} f_1^{(v)} f_2^{(w)}, -l \geq u+k-j, -m \geq j+v-u, k \geq h+j, v \geq u+w. \quad (48)$$

$$f_2^{(u)} f_1^{(v)} f_2^{(w)} 1_{(m,l)} e_1^{(h)} e_2^{(k)} e_1^{(j)}, -l \leq h-k-w, -m \leq w-v-h, k \geq h+j, v \geq u+w. \quad (49)$$

$$e_1^{(h)} e_2^{(k)} 1_{(m,l)} f_2^{(v)} f_1^{(w)}, -l \geq v+k, -m \geq w-v+h-k, h \leq k, v \geq w. \quad (50)$$

$$f_1^{(u)} f_2^{(v)} 1_{(m,l)} e_2^{(k)} e_1^{(j)}, -l \leq -k-v, -m \leq v-u+k-j, j \leq k, v \geq u. \quad (51)$$

$$e_1^{(h)} e_2^{(k)} e_1^{(j)} 1_{(m,l)} f_2^{(v)}, -l \geq v+k-j, k \geq h+j, v \geq 0. \quad (52)$$

$$f_2^{(v)} 1_{(m,l)} e_1^{(h)} e_2^{(k)} e_1^{(j)}, -l \leq -v+h-k, k \geq h+j, v \geq 0. \quad (53)$$

$$e_1^{(h)} e_2^{(k)} e_1^{(j)} 1_{(m,l)} f_2^{(v)} f_1^{(w)}, -l \geq v+k-j, -m \geq w-v+j, k \geq h+j, v \geq w. \quad (54)$$

$$f_1^{(u)} f_2^{(v)} 1_{(m,l)} e_1^{(h)} e_2^{(k)} e_1^{(j)}, -l \leq -v+h-k, -m \leq v-u-h, k \geq h+j, v \geq u. \quad (55)$$

$$e_2^{(k)} 1_{(m,l)} f_1^{(v)}, k \leq 0, v \leq 0. \quad (56)$$

$$e_2^{(k)} 1_{(m,l)} f_1^{(v)} f_2^{(w)}, -l \geq w-v+k, k \leq 0, v \geq w. \quad (57)$$

$$f_2^{(u)} f_1^{(v)} 1_{(m,l)} e_2^{(k)}, -l \leq v-u-k, k \leq 0, v \geq u. \quad (58)$$

$$e_1^{(h)} e_2^{(k)} 1_{(m,l)} f_1^{(v)}, -m \geq v+h-k, h \leq k, v \geq 0. \quad (59)$$

$$f_1^{(v)} 1_{(m,l)} e_2^{(k)} e_1^{(j)}, -m \leq k-j-v, j \leq k, v \geq 0. \quad (60)$$

$$e_1^{(h)} e_2^{(k)} 1_{(m,l)} f_1^{(v)} f_2^{(w)}, -l \geq w-v+k, -m \geq v+h-k, h \leq k, v \geq w. \quad (61)$$

$$f_2^{(u)} f_1^{(v)} 1_{(m,l)} e_2^{(k)} e_1^{(j)}, -l \leq v-u-k, -m \leq -v+k-j, j \leq k, v \geq u. \quad (62)$$

$$e_1^{(h)} e_2^{(k)} e_1^{(j)} 1_{(m,l)} f_1^{(v)}, -m \geq j+v, k \geq h+j, v \geq 0. \quad (63)$$

$$f_1^{(v)} 1_{(m,l)} e_1^{(h)} e_2^{(k)} e_1^{(j)}, -m \leq -v-h, k \geq h+j, v \geq 0. \quad (64)$$

$$e_1^{(h)} e_2^{(k)} e_1^{(j)} 1_{(m,l)} f_1^{(v)} f_2^{(w)}, -l \geq w - v + k - j, -m \geq j + v, k \geq h + j, v \geq w. \quad (65)$$

$$f_2^{(u)} f_1^{(v)} 1_{(m,l)} e_1^{(h)} e_2^{(k)} e_1^{(j)}, -l \leq v - u + h - k, -m \leq -v - h, k \geq h + j, v \geq u. \quad (66)$$

proof. We only prove **Part (1)**, the proof of **Part (2)** is similar. We want to compute the image of these elements under the map $\dot{\mathbf{U}} \rightarrow V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, which is given by $u \mapsto u(\xi_{(-s,-t)} \otimes \eta_{(a,b)})$.

(1) For the element $e_1^{(k)} 1_{(l,m)} f_2^{(u)} f_1^{(v)} f_2^{(w)}$, its image is zero unless $l + 2v - (u + w) = a - s, m + 2(u + w) - v = b - t$, if $-l \geq v + k - u, v \geq u + w$, then we get $v \geq a + w + k - s$, then we get $e_1^{(k)} 1_{(l,m)} f_2^{(u)} f_1^{(v)} f_2^{(w)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) =$

$$\sum_{0 \leq p \leq k, v} v^{p(k-p-s)} \begin{bmatrix} a + p - v + w \\ p \end{bmatrix} e_1^{(k-p)} \xi_{(-s,-t)} \otimes f_2^{(u)} f_1^{(v-p)} f_2^{(w)} \eta_{(a,b)}$$

Let A denote the degree of the coefficient, then $A = p(k - p - s) + p(a - v + w)$. If $v \geq a + w + k - s$, $A \leq 0$ and $A = 0$ if and only if $p = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $e_1^{(k)} 1_{(l,m)} f_2^{(u)} f_1^{(v)} f_2^{(w)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_1^{(k)} \diamond \theta_2^{(u)} \theta_1^{(v)} \theta_2^{(w)})_{(-s,-t),(a,b)}$.

(2) For the element $f_2^{(u)} f_1^{(v)} f_2^{(w)} 1_{(l,m)} e_1^{(k)}$, its image is zero unless $l - 2k = a - s, m = b - t$, if $-l \leq w - k - v, v \geq u + w$, then we get $v \leq a + w + k - s$, then we get $f_2^{(u)} f_1^{(v)} f_2^{(w)} 1_{(l,m)} e_1^{(k)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) =$

$$\sum_{0 \leq p \leq k, v} v^{p(v-a-w-p)} \begin{bmatrix} s + p - k \\ p \end{bmatrix} e_1^{(k-p)} \xi_{(-s,-t)} \otimes f_2^{(u)} f_1^{(v-p)} f_2^{(w)} \eta_{(a,b)}$$

Let B denote the degree of the coefficient, then $B = p(v - a - w - p) + p(s - k)$. If $v \leq a + w + k - s$, $B \leq 0$ and $B = 0$ if and only if $p = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $f_2^{(u)} f_1^{(v)} f_2^{(w)} 1_{(l,m)} e_1^{(k)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_1^{(k)} \diamond \theta_2^{(u)} \theta_1^{(v)} \theta_2^{(w)})_{(-s,-t),(a,b)}$.

(3) For the element $e_2^{(h)} e_1^{(k)} 1_{(l,m)} f_2^{(u)} f_1^{(v)} f_2^{(w)}$, its image is zero unless $l + 2v - (u + w) = a - s, m + 2(u + w) - v = b - t$, if $-l \geq v + k - u, -m \geq u + h - k, h \leq k, v \geq u + w$, then we get $v \geq a + w + k - s, u + 2w \geq b + v + h - k - t$, then we get $e_2^{(h)} e_1^{(k)} 1_{(l,m)} f_2^{(u)} f_1^{(v)} f_2^{(w)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) =$

$$\sum_{\substack{0 \leq p \leq k, v \\ 0 \leq q \leq h \\ 0 \leq t' \leq q, u}} v^{p(k-p-s)+q(h-q-t-k+p)} \begin{bmatrix} a + p - v + w \\ p \end{bmatrix} \begin{bmatrix} b + q - u - 2w + v - p \\ t' \end{bmatrix} \begin{bmatrix} b + q - t' - w \\ q - t' \end{bmatrix} e_2^{(h-q)} e_1^{(k-p)} \xi_{(-s,-t)} \otimes f_2^{(u-t')} f_1^{(v-p)} f_2^{(w-q+t')} \eta_{(a,b)}$$

Let C denote the degree of the coefficient, then $C = p(k - p - s) + q(h - q - t - k + p) + p(a - v + w) + t'(b + v + q - u - 2w - p - t') + (q - t')(b - w)$. If

$$v \geq a + w + k - s, u + 2w \geq b + v + h - k - t, \text{ then } C = -p^2 + p(a + w - v + k - s) + q(h - k) + q(b - w - q + p - t) + t'(v - u - w + q - p - t') \leq -p^2 + p(a + w - v + k - s) + q(h - k) + q(u + w - v - h + k - q + p) + t'(v - u - w + q - p - t') = -p^2 + p(a + w - v + k - s) + (t' - q)(v - u - w) + q(-q + p) + t'(q - p) - t'^2 \leq -p^2 + q(-q + p) + t'(q - p) - t'^2 = \begin{cases} -(q - p)^2 - qp - t'^2 + (q - p)t' \leq 0 & \text{if } p \geq q \\ -p^2 - t'^2 + (t' - q)(q - p) \leq 0 & \text{if } p \leq q \end{cases}$$

So in this case we get $C \leq 0$ and $C = 0$ if and only if $p = q = t' = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $e_2^{(h)} e_1^{(k)} 1_{(l,m)} f_2^{(u)} f_1^{(v)} f_2^{(w)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_2^{(h)} \theta_1^{(k)} \diamond \theta_2^{(u)} \theta_1^{(v)} \theta_2^{(w)})_{(-s,-t),(a,b)}$.

(4) For the element $f_2^{(u)} f_1^{(v)} f_2^{(w)} 1_{(l,m)} e_2^{(h)} e_1^{(k)}$, its image is zero unless $l - 2k + h = a - s, m - 2h + k = b - t$, if $-l \leq w - v + h - k, -m \leq -w - h, h \leq k, v \geq u + w$, then we get $v \leq a + w + k - s, w \leq b + h - k - t$. then we get $f_2^{(u)} f_1^{(v)} f_2^{(w)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) =$

$$\sum_{\substack{0 \leq r \leq w \\ 0 \leq p \leq k, v \\ 0 \leq q \leq u}} v^{r(w-r-b)+p(v-a-w-p+r)+q(u-q-b+2w-2r-v+p)} \begin{bmatrix} s+p-k \\ p \end{bmatrix} \begin{bmatrix} t+r-h+k \\ r \end{bmatrix} \begin{bmatrix} t-h+k+q-p+r \\ q \end{bmatrix} e_2^{(h-r-q)} e_1^{(k-p)} \xi_{(-s,-t)} \otimes f_2^{(u-q)} f_1^{(v-p)} f_2^{(w-r)} \eta_{(a,b)}$$

Let D denote the degree of the coefficient, then $D = r(w-r-b)+p(v-a-w-p+r)+q(u-q-b+2w-2r-v+p)+r(t-h+k)+p(s-k)+q(t-h+k-p+r)$. If $v \leq a + w + k - s, w \leq b + h - k - t$, then $D = -p^2 + pr + p(v - a - w + s - k) - r^2 + (r + q)(w - b + t - h + k) + q(u + w - v) - q^2 - qr = -(p-r)^2 - pr - q^2 - qr + p(v - a - w + s - k) + (r + q)(w - b + t - h + k) + q(u + w - v)$. then we get $D \leq 0$ and $D = 0$ if and only if $r = p = q = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $f_2^{(u)} f_1^{(v)} f_2^{(w)} 1_{(l,m)} e_2^{(h)} e_1^{(k)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_2^{(h)} \theta_1^{(k)} \diamond \theta_2^{(u)} \theta_1^{(v)} \theta_2^{(w)})_{(-s,-t),(a,b)}$.

(5) is obviously contained in (7).

(6) For the element $f_2^{(u)} f_1^{(v)} f_2^{(w)} 1_{(l,m)} e_1^{(k)} e_2^{(j)}$, its image is zero unless $l - 2k + j = a - s, m - 2j + k = b - t$, if $-l \leq w - v - k, -m \leq -w + k - j, k \geq h + j, v \geq u + w$, then we get $v \leq a + w + k - s - j, w \leq b + j - t$. then we get $f_2^{(u)} f_1^{(v)} f_2^{(w)} 1_{(l,m)} e_1^{(k)} e_2^{(j)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) =$

$$\sum_{\substack{0 \leq r \leq w \\ 0 \leq p \leq k, v \\ 0 \leq q \leq u, j}} v^{r(w-r-b)+p(v-a-w-p+r)+q(u-q-b+2w-2r-v+p)} \begin{bmatrix} s+p-k+j-r \\ p \end{bmatrix} \begin{bmatrix} t-j+r \\ r \end{bmatrix} \begin{bmatrix} t+q-j+r \\ q \end{bmatrix} e_1^{(k-p)} e_2^{(j-r-q)} \xi_{(-s,-t)} \otimes f_2^{(u-q)} f_1^{(v-p)} f_2^{(w-r)} \eta_{(a,b)}$$

Let F denote the degree of the coefficient, then $F = r(w - r - b) + p(v -$

$a-w-p+r)+q(u-q-b+2w-2r-v+p)+r(t-j)+p(s-k+j-r)+q(t-j+r)$.
 If $v \leq a+w+k-s-j$, $w \leq b+j-t$, then $F = -p^2 + p(v-a-w+s+j-k) - r^2 + r(w-b+t-j) + q(u+w-v) - q^2 + q(w-b+t-j) + q(p-r) = -(q-p)^2 - qp - r^2 - qr + p(v-a-w+s+j-k) + (r+q)(w-b+t-j) + q(u+w-v)$.
 then we get $F \leq 0$ and $F = 0$ if and only if $r = p = q = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $f_2^{(u)} f_1^{(v)} f_2^{(w)} 1_{(l,m)} e_1^{(k)} e_2^{(j)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_1^{(k)} \theta_2^{(j)} \diamond \theta_2^{(u)} \theta_1^{(v)} \theta_2^{(w)})_{(-s,-t),(a,b)}$.

(7) For the element $e_2^{(h)} e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_2^{(u)} f_1^{(v)} f_2^{(w)}$, its image is zero unless $l+2v-(u+w) = a-s$, $m+2(u+w)-v = b-t$, if $-l \geq v+k-j-u$, $-m \geq u+j$, $k \geq h+j$, $v \geq u+w$, then we get $v \geq a+w+k-s-j$, $u+2w \geq b+v+j-t$, $w \geq b+j-t$. then we get $e_2^{(h)} e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_2^{(u)} f_1^{(v)} f_2^{(w)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) =$

$$\sum_{\substack{0 \leq p \leq k, v \\ 0 \leq r \leq j, 0 \leq d \leq u, r \\ 0 \leq q \leq h, 0 \leq t' \leq q, u-d}} v^{r(j-r-t)+p(k-p-s-j+r)+q(h-q-t-k+p+2j-2r)} \begin{bmatrix} b+r-u-2w+v \\ d \end{bmatrix}$$

$$\begin{bmatrix} b+r-d-w \\ r-d \end{bmatrix} \begin{bmatrix} a+p-v+w-r+d \\ p \end{bmatrix} \begin{bmatrix} b+q-u-2w+v-p+2r-d \\ t' \end{bmatrix}$$

$$\begin{bmatrix} b+q-t'-w+r-d \\ q-t' \end{bmatrix} e_2^{(h-q)} e_1^{(k-p)} e_2^{(j-r)} \xi_{(-s,-t)} \otimes f_2^{(u-d-t')} f_1^{(v-p)} f_2^{(w-r+d-q+t')} \eta_{(a,b)}$$

Let E denote the degree of the coefficient, then $E = r(j-r-t)+p(k-p-s-j+r)+q(h-q-t-k+p+2j-2r)+d(b+r-d-u-2w+v)+(r-d)(b-w)+p(a-v+w-r+d)+t'(b+v+q-u-2w-p-t'+2r-d)+(q-t')(b-w+r-d)$.
 If $v \geq a+w+k-s-j$, $u+2w \geq b+v+j-t$, then $E \leq -p^2 + pd + p(a+w-v+k-s-j) + q(h+j-k) + q(-q+p-t+j-2r) + t'(q-t'-j+t+2r-d-p) + (q-t')(t-j+r-d) + r(j-r-t) + d(r-d+t-j) + (r-d)(t-j) \leq -p^2 + pd - r^2 + d(r-d) + p(a+w-v+k-s-j) + q(h+j-k) + q(-q+p-r-d) + t'(q-t'+r-p) \leq -p^2 + pd - r^2 + d(r-d) + q(-q+p-r-d) + t'(q-t'+r-p)$.
 If $q \geq p$, then $E \leq -(p-d)^2 - pd + r(d-r) + (t'-q)(q-p+r) - t'^2 - qd \leq 0$; If $q < p$, then $E \leq -p^2 + pd - r^2 + d(r-d) + q(-q+p-r-d) + t'(q-t'+r-p) =$

$$\begin{cases} -(p-d)^2 - pd + r(d-r) - t'^2 + (q-p)t' + r(t'-q) - q^2 + q(p-d) \leq 0 & \text{if } p \leq d \\ -t'^2 + (q-p)t' + r(t'-q) - q^2 + (p-q)(d-p) + r(d-r) - d^2 < 0 & \text{if } p > d \end{cases}$$

So in this case we get $E \leq 0$ and $E = 0$ if and only if $p = r = d = q = t' = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $e_2^{(h)} e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_2^{(u)} f_1^{(v)} f_2^{(w)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_2^{(h)} \theta_1^{(k)} \theta_2^{(j)} \diamond \theta_2^{(u)} \theta_1^{(v)} \theta_2^{(w)})_{(-s,-t),(a,b)}$.

(8) For the element $f_2^{(u)} f_1^{(v)} f_2^{(w)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} e_2^{(j)}$, its image is zero unless $l-2k+(h+j) = a-s$, $m-2(h+j)+k = b-t$, if $-l \leq w-v+h-k$, $-m \leq -w-h$, $k \geq h+j$, $v \geq u+w$, then we get $v \leq a+w+k-s-j$, $w \leq b+j+(j+h-k)-t$. then we get $f_2^{(u)} f_1^{(v)} f_2^{(w)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} e_2^{(j)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) =$

$$\sum_{\substack{0 \leq r \leq w \\ 0 \leq d \leq r, h \leq p \leq v \\ 0 \leq q \leq u, 0 \leq t' \leq q, h-d}} v^{r(w-r-b)+p(v-a-w-p+r)+q(u-q-b+2w-2r-v+p)} \begin{bmatrix} s+p-k+j-r+d \\ p \end{bmatrix} \begin{bmatrix} t-h+k-2j+r \\ d \end{bmatrix} \begin{bmatrix} t+r-d-j \\ r-d \end{bmatrix} \begin{bmatrix} t-h+k+q-p-2j+2r-d \\ t' \end{bmatrix} \begin{bmatrix} t+q-t'-j+r-d \\ q-t' \end{bmatrix} e_2^{(h-d-t')} e_1^{(k-p)} e_2^{(j-r+d-q+t')} \xi_{(-s,-t)} \otimes f_2^{(u-q)} f_1^{(v-p)} f_2^{(w-r)} \eta_{(a,b)}$$

Let G denote the degree of the coefficient, then $G = r(w-r-b) + p(v-a-w-p+r) + q(u-q-b+2w-2r-v+p) + p(s-k+j-r+d) + d(t-h+k-2j+r-d) + (r-d)(t-j) + t'(t-h+k+q-p-2j+2r-d-t') + (q-t')(t-j+r-d)$. If $v \leq a+w+k-s-j$, $w \leq b+j+(j+h-k)-t$, then $G \leq -p^2 + pd - r^2 + d(r-d) + q(-q-r+p-d) + t'(q-p+r-t') + p(v-a-w-s+j-k) + (r-d)(w-b+t-j) + q(u+w-v) + (t'-q)(k-j-h) \leq -p^2 + pd - r^2 + d(r-d) + q(-q-r+p-d) + t'(q-p+r-t')$.

then from the discusstion in (5) we get $G \leq 0$ and $G = 0$ if and only if $r = d = p = q = t' = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $f_2^{(u)} f_1^{(v)} f_2^{(w)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} e_2^{(j)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_2^{(h)} \theta_1^{(k)} \theta_2^{(j)} \diamond \theta_2^{(u)} \theta_1^{(v)} \theta_2^{(w)})_{(-s,-t),(a,b)}$.

(9) For the element $e_1^{(k)} 1_{(l,m)} f_1^{(u)} f_2^{(v)} f_1^{(w)}$, its image is zero unless $l+2(u+w)-v = a-s$, $m+2v-(u+w) = b-t$, if $-l \geq k+u$, $v \geq u+w$, then we get $u+2w \geq a+v+k-s$, then we get

$$e_1^{(k)} 1_{(l,m)} f_1^{(u)} f_2^{(v)} f_1^{(w)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) = \sum_{\substack{0 \leq p \leq k \\ 0 \leq r \leq p, u}} v^{p(k-p-s)} \begin{bmatrix} a+p-u-2w+v \\ r \end{bmatrix} \begin{bmatrix} a+p-r-w \\ p-r \end{bmatrix} e_1^{(k-p)} \xi_{(-s,-t)} \otimes f_1^{(u-r)} f_2^{(v)} f_1^{(w-p+r)} \eta_{(a,b)}$$

Let H denote the degree of the coefficient, then $H = p(k-p-s) + r(a-u-2w+v+p-r) + (p-r)(a-w)$. If $u+2w \geq a+v+k-s$, $H \leq p(k-p-s) + r(a-u-2w+v+p-r) + (p-r)(s-k) \leq -(p-r)^2 - pr + r(a+v-u-2w+k-s) \leq 0$ and $H = 0$ if and only if $p = r = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $e_1^{(k)} 1_{(l,m)} f_1^{(u)} f_2^{(v)} f_1^{(w)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_1^{(k)} \diamond \theta_1^{(u)} \theta_2^{(v)} \theta_1^{(w)})_{(-s,-t),(a,b)}$.

(10) For the element $f_1^{(u)} f_2^{(v)} f_1^{(w)} 1_{(l,m)} e_1^{(k)}$, its image is zero unless $l-2k = a-s$, $m = b-t$, if $-l \leq -w-k$, $v \geq u+w$, then we get $w \leq a+k-s$, then we get $f_1^{(u)} f_2^{(v)} f_1^{(w)} 1_{(l,m)} e_1^{(k)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) = \sum_{\substack{0 \leq r \leq w \\ 0 \leq q \leq u}} v^{r(w-r-a)+q(u-q-a+2w-2r-v)}$

$$\begin{bmatrix} s+r-k \\ r \end{bmatrix} \begin{bmatrix} s+q-k+r \\ q \end{bmatrix} e_1^{(k-r)} \xi_{(-s,-t)} \otimes f_1^{(u-q)} f_2^{(v)} f_1^{(w-r)} \eta_{(a,b)}$$

Let I denote the degree of the coefficient, then $I = r(w - r - a) + q(u - q - a + 2w - 2r - v) + q(s - k + r) + r(s - k)$. If $w \leq a + k - s$, $I = -r^2 - qr - q^2 + (r + q)(w - a + s - k) + q(u + w - v) \leq 0$ and $I = 0$ if and only if $q = r = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $f_1^{(u)} f_2^{(v)} f_1^{(w)} 1_{(l,m)} e_1^{(k)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_1^{(k)} \diamond \theta_1^{(u)} \theta_2^{(v)} \theta_1^{(w)})_{(-s,-t),(a,b)}$.

(11) For the element $e_2^{(h)} e_1^{(k)} 1_{(l,m)} f_1^{(u)} f_2^{(v)} f_1^{(w)}$, its image is zero unless $l + 2(u + w) - v = a - s, m + 2v - (u + w) = b - t$, if $-l \geq u + k, -m \geq v - u + h - k, h \leq k, v \geq u + w$, then we get $v \geq b + w + h - k - t, u + 2w \geq a + v + k - s$, then we get $e_2^{(h)} e_1^{(k)} 1_{(l,m)} f_1^{(u)} f_2^{(v)} f_1^{(w)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) =$

$$\sum_{\substack{0 \leq p \leq k \\ 0 \leq q \leq p, u \\ 0 \leq r \leq h}} v^{p(k-p-s)+r(h-r-t-k+p)} \begin{bmatrix} a + p - u - 2w + v \\ q \end{bmatrix} \begin{bmatrix} a + p - q - w \\ p - q \end{bmatrix}$$

$$\begin{bmatrix} b + r - v + w - p + q \\ r \end{bmatrix} e_2^{(h-r)} e_1^{(k-p)} \xi_{(-s,-t)} \otimes f_1^{(u-q)} f_2^{(v-r)} f_1^{(w-p+q)} \eta_{(a,b)}$$

Let J denote the degree of the coefficient, then $J = p(k - p - s) + r(h - r - t - k + p) + q(a + p - u - 2w + v - q) + (p - q)(a - w) + r(b - v + w - p + q)$. If $v \geq b + w + h - k - t, u + 2w \geq a + v + k - s$, then $J = p(a - w + k - s - p) + r(h - k + b - v + w - r - t + q) + q(v - u - w + p - q) \leq (q - p)(v - u - w) + p(q - p) - (r - q)^2 - rq \leq 0$

So in this case we get $J \leq 0$ and $J = 0$ if and only if $p = q = r = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $e_2^{(h)} e_1^{(k)} 1_{(l,m)} f_1^{(u)} f_2^{(v)} f_1^{(w)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_2^{(h)} \theta_1^{(k)} \diamond \theta_1^{(u)} \theta_2^{(v)} \theta_1^{(w)})_{(-s,-t),(a,b)}$.

(12) For the element $f_1^{(u)} f_2^{(v)} f_1^{(w)} 1_{(l,m)} e_2^{(h)} e_1^{(k)}$, its image is zero unless $l - 2k + h = a - s, m - 2h + k = b - t$, if $-l \leq -w + h - k, -m \leq w - v - h, h \leq k, v \geq u + w$, then we get $v \leq b + w + h - k - t, w \leq a + k - s$. then we get

$$f_1^{(u)} f_2^{(v)} f_1^{(w)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) =$$

$$\sum_{\substack{0 \leq r \leq w \\ 0 \leq p \leq h, v \\ 0 \leq q \leq u}} v^{r(w-r-a)+p(v-b-w-p+r)+q(u-q-a+2w-2r-v+p)} \begin{bmatrix} s + r - k \\ r \end{bmatrix} \begin{bmatrix} s + q - k + r \\ q \end{bmatrix}$$

$$\begin{bmatrix} t - h + k - r + p \\ p \end{bmatrix} e_2^{(h-p)} e_1^{(k-r-q)} \xi_{(-s,-t)} \otimes f_1^{(u-q)} f_2^{(v-p)} f_1^{(w-r)} \eta_{(a,b)}$$

Let K denote the degree of the coefficient, then $K = r(w - r - a) + p(v - b - w - p + r) + q(u - q - a + 2w - 2r - v + p) + p(t - h + k - r) + r(s - k) + q(s - k + r)$. If $v \leq b + w + h - k - t, w \leq a + k - s$, then $K = -p^2 + p(v - b - w + t - h + k) - r^2 + (r + q)(w - a + s - k) + q(u + w - v) - q^2 - qr + qp = -(p - q)^2 - pq -$

$r^2 - qr + p(v - b - w + t - h + k) + (r + q)(w - a + s - k) + q(u + w - v) \leq 0$. then we get $K \leq 0$ and $K = 0$ if and only if $r = p = q = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $f_1^{(u)} f_2^{(v)} f_1^{(w)} 1_{(l,m)} e_2^{(h)} e_1^{(k)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_2^{(h)} \theta_1^{(k)} \diamond \theta_1^{(u)} \theta_2^{(v)} \theta_1^{(w)})_{(-s,-t),(a,b)}$.

(13) is obviously contained in (15).

(14) For the element $f_1^{(u)} f_2^{(v)} f_1^{(w)} 1_{(l,m)} e_1^{(k)} e_2^{(j)}$, its image is zero unless $l - 2k + j = a - s, m - 2j + k = b - t$, if $-l \leq -w - k, -m \leq w - v + k - j, k \geq j, v \geq u + w$, then we get $v \leq b + w + j - t, w \leq a + k - s - j$. then we get $f_1^{(u)} f_2^{(v)} f_1^{(w)} 1_{(l,m)} e_1^{(k)} e_2^{(j)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) =$

$$\sum_{\substack{0 \leq r \leq w \\ 0 \leq p \leq v, 0 \leq q \leq u}} v^{r(w-r-a)+p(v-b-w-p+r)+q(u-q-a+2w-2r-v+p)} \begin{bmatrix} s + q - k + r + j - p \\ q \end{bmatrix} \begin{bmatrix} s + r - k + j \\ r \end{bmatrix} \begin{bmatrix} t + p - j \\ p \end{bmatrix} e_1^{(k-r-q)} e_2^{(j-p)} \xi_{(-s,-t)} \otimes f_1^{(u-q)} f_2^{(v-p)} f_1^{(w-r)} \eta_{(a,b)}$$

Let M denote the degree of the coefficient, then $M = r(w - r - a) + p(v - b - w - p + r) + q(u - q - a + 2w - 2r - v + p) + r(s - k + j) + p(t - j) + q(s - k + r + j - p)$. If $v \leq b + w + j - t, w \leq a + k - s - j$, then $M = -r^2 + r(w - a + s + j - k) + p(-p + r) + p(v - b - w + t - j) + q(u + w - v) + q(w - a + s + j - k) + q(-q - r) \leq -(r - p)^2 - pr - q^2 - qr$.

then we get $M \leq 0$ and $M = 0$ if and only if $r = p = q = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $f_1^{(u)} f_2^{(v)} f_1^{(w)} 1_{(l,m)} e_1^{(k)} e_2^{(j)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_1^{(k)} \theta_2^{(j)} \diamond \theta_1^{(u)} \theta_2^{(v)} \theta_1^{(w)})_{(-s,-t),(a,b)}$.

(15) For the element $e_2^{(h)} e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_1^{(u)} f_2^{(v)} f_1^{(w)}$, its image is zero unless $l + 2(u + w) - v = a - s, m + 2v - (u + w) = b - t$, if $-l \geq u + k - j, -m \geq v - u + j, k \geq h + j, v \geq u + w$, then we get $v \geq b + w + j - t, u + 2w \geq a + v + k - s - j$. then we get $e_2^{(h)} e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_1^{(u)} f_2^{(v)} f_1^{(w)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) =$

$$\sum_{\substack{0 \leq f \leq j, 0 \leq p \leq k \\ 0 \leq q \leq p, u, 0 \leq r \leq h}} v^{f(j-f-t)+p(k-p-s-j+f)+r(h-r-t-k+p+2j-2f)} \begin{bmatrix} b + f - v + w \\ f \end{bmatrix} \begin{bmatrix} a + p - q - w \\ p - q \end{bmatrix} \begin{bmatrix} a + p - u - 2w + v - f \\ q \end{bmatrix} \begin{bmatrix} b + r - v + f + w - p + q \\ r \end{bmatrix} e_2^{(h-r)} e_1^{(k-p)} e_2^{(j-f)} \xi_{(-s,-t)} \otimes f_1^{(u-q)} f_2^{(v-f-r)} f_1^{(w-p+q)} \eta_{(a,b)}$$

Let L denote the degree of the coefficient, then $L = f(j - f - t) + p(k - p - s - j + f) + r(h - r - t - k + p + 2j - 2f) + f(b - v + w) + q(a + p - u - 2w + v - f - q) + (p - q)(a - w) + r(b - v + f + w - p + q)$. If $v \geq b + w + j - t, u + 2w \geq a + v + k - s - j$, then $L \leq -f^2 + f(b - v + w + j - t) + p(u + w - v - p + f) + q(p + v - u - w - f - q) + r(b - v + w + j - t - r - f + q) \leq -p^2 + pf + qp - qf - q^2 - f^2 - r^2 - rf + qr \leq$

$$\begin{cases} p(q-p) + (r+f)(p-f) - qf - q^2 - r^2 \leq 0 & \text{if } p \leq f \\ (q-p)(p-f) - f^2 - rf - (q-r)^2 - qr \leq 0 & \text{if } p > f \end{cases}$$

So in this case we get $L \leq 0$ and $L = 0$ if and only if $p = r = f = q = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $e_2^{(h)} e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_1^{(u)} f_2^{(v)} f_1^{(w)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_2^{(h)} \theta_1^{(k)} \theta_2^{(j)} \diamond \theta_1^{(u)} \theta_2^{(v)} \theta_1^{(w)})_{(-s,-t),(a,b)}$.

(16) For the element $f_1^{(u)} f_2^{(v)} f_1^{(w)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} e_2^{(j)}$, its image is zero unless $l - 2k + (h + j) = a - s, m - 2(h + j) + k = b - t$, if $-l \leq h - k - w, -m \leq w - v - h, k \geq h + j, v \geq u + w$, then we get $v \leq b + w + j + (j + h - k) - t, w \leq a + k - s - j$. then we get $f_1^{(u)} f_2^{(v)} f_1^{(w)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} e_2^{(j)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) =$

$$\sum_{\substack{0 \leq r \leq w \\ 0 \leq q \leq u \\ 0 \leq p \leq v \\ 0 \leq t' \leq p, h}} v^{r(w-r-a)+p(v-b-w-p+r)+q(u-q-a+2w-2r-v+p)} \begin{bmatrix} s + r - k + j \\ r \end{bmatrix}$$

$$\begin{bmatrix} s + q - k + r + j - p + t' \\ q \end{bmatrix} \begin{bmatrix} t + p - t' - j \\ p - t' \end{bmatrix} \begin{bmatrix} t - h + k - r - 2j + p \\ t' \end{bmatrix}$$

$$e_2^{(h-t')} e_1^{(k-r-q)} e_2^{(j-p+t')} \xi_{(-s,-t)} \otimes f_1^{(u-q)} f_2^{(v-p)} f_1^{(w-r)} \eta_{(a,b)}$$

Let N denote the degree of the coefficient, then $N = r(w - r - a) + p(v - b - w - p + r) + q(u - q - a + 2w - 2r - v + p) + r(s - k + j) + t'(t - h + k - r - 2j + p - t') + (p - t')(t - j) + q(s - k + r + j - p + t')$. If $v \leq b + w + j + (j + h - k) - t, w \leq a + k - s - j$, then $N \leq -r^2 + r(w - a + s - k + j) + p(-p + r) + p(v - b - w + t - j) + t'(k - j - h) + t'(-r + p - t') + q(u + w - v) + q(w - a + s - k + j) + q(-q - r + t') \leq -r^2 - p^2 + pr + q(-q - r + t') + t'(-r + p - t') =$

$$\begin{cases} -(p - r)^2 - pr + t'(p - r) - (t' - q)^2 - qt' - qr \leq 0 & \text{if } p \leq r \\ -r^2 + (t' - p)(p - r) - (t' - q)^2 - qt' - qr \leq 0 & \text{if } p > r \end{cases}$$

Then we get $N \leq 0$ and $N = 0$ if and only if $p = r = q = t' = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $f_1^{(u)} f_2^{(v)} f_1^{(w)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} e_2^{(j)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_2^{(h)} \theta_1^{(k)} \theta_2^{(j)} \diamond \theta_1^{(u)} \theta_2^{(v)} \theta_1^{(w)})_{(-s,-t),(a,b)}$.

(17) For the element $e_2^{(h)} e_1^{(k)} 1_{(l,m)} f_1^{(v)} f_2^{(w)}$, its image is zero unless $l + 2v - w = a - s, m + 2w - v = b - t$, if $-l \geq v + k, -m \geq w - v + h - k, h \leq k, v \geq w$, then we get $v \geq a + w + k - s, w \geq b + h - k - t$, then we get $e_2^{(h)} e_1^{(k)} 1_{(l,m)} f_1^{(v)} f_2^{(w)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) =$

$$\sum_{\substack{0 \leq p \leq k, v \\ 0 \leq q \leq h, w}} v^{p(k-p-s)+q(h-q-t-k+p)}$$

$$\begin{bmatrix} a + p - v + w \\ p \end{bmatrix} \begin{bmatrix} b + q - w \\ q \end{bmatrix} e_2^{(h-q)} e_1^{(k-p)} \xi_{(-s,-t)} \otimes f_1^{(v-p)} f_2^{(w-q)} \eta_{(a,b)}$$

Let O denote the degree of the coefficient, then $O = p(k - p - s) + q(h - q - t - k + p) + p(a - v + w) + q(b - w)$. If $v \geq a + w + k - s, w \geq b + h - k - t$, then $O = -p^2 + qp - q^2 + p(a + w - v + k - s) + q(h - k + b - w - t) \leq 0$.

So in this case we get $O \leq 0$ and $O = 0$ if and only if $p = q = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $e_2^{(h)} e_1^{(k)} 1_{(l,m)} f_1^{(v)} f_2^{(w)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_2^{(h)} \theta_1^{(k)} \diamond \theta_1^{(v)} \theta_2^{(w)})_{(-s,-t),(a,b)}$.

(18) For the element $f_2^{(u)} f_1^{(v)} 1_{(l,m)} e_1^{(k)} e_2^{(j)}$, its image is zero unless $l - 2k + j = a - s, m - 2j + k = b - t$, if $-l \leq -v - k, -m \leq v - u + k - j, k \geq j, v \geq u$, then we get $v \leq a + k - s - j, u \leq b + v + j - t$. then we get $f_2^{(u)} f_1^{(v)} 1_{(l,m)} e_1^{(k)} e_2^{(j)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) = \sum_{\substack{0 \leq p \leq k, v \\ 0 \leq q \leq u, j}} v^{p(v-a-p)+q(u-q-b-v+p)}$

$$\begin{bmatrix} s + p - k + j \\ p \end{bmatrix} \begin{bmatrix} t + q - j \\ q \end{bmatrix} e_1^{(k-p)} e_2^{(j-q)} \xi_{(-s,-t)} \otimes f_2^{(u-q)} f_1^{(v-p)} \eta_{(a,b)}$$

Let P denote the degree of the coefficient, then $P = p(v - a - p) + q(u - q - b - v + p) + p(s - k + j) + q(t - j)$. If $v \leq a + k - s - j, u \leq b + v + j - t$, then $P = -p^2 + p(v - a + s + j - k) - q^2 + q(u - v - b + t - j) + qp = -(q - p)^2 - qp + p(v - a + s + j - k) + q(u - v - b + t - j) \leq 0$.

then we get $P \leq 0$ and $P = 0$ if and only if $p = q = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $f_2^{(u)} f_1^{(v)} 1_{(l,m)} e_1^{(k)} e_2^{(j)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_1^{(k)} \theta_2^{(j)} \diamond \theta_2^{(u)} \theta_1^{(v)})_{(-s,-t),(a,b)}$.

(19) For the element $e_2^{(h)} e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_1^{(v)}$, its image is zero unless $l + 2v = a - s, m - v = b - t$, if $-l \geq v + k - j, k \geq h + j, v \geq 0$, then we get $v \geq a + w + k - s - j$. then we get $e_2^{(h)} e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_1^{(v)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) = \sum_{0 \leq p \leq k, v}$

$$v^{p(k-p-s-j)} \begin{bmatrix} a + p - v + w \\ p \end{bmatrix} e_2^{(h)} e_1^{(k-p)} e_2^{(j)} \xi_{(-s,-t)} \otimes f_1^{(v-p)} \eta_{(a,b)}$$

Let R denote the degree of the coefficient, then $R = p(k - p - s - j) + p(a - v + w)$. If $v \geq a + w + k - s - j$, then $R \leq -p^2 + p(a + w - v + k - s - j) \leq 0$.

So in this case we get $R \leq 0$ and $R = 0$ if and only if $p = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $e_2^{(h)} e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_1^{(v)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_2^{(h)} \theta_1^{(k)} \theta_2^{(j)} \diamond \theta_1^{(v)})_{(-s,-t),(a,b)}$.

(20) For the element $f_1^{(v)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} e_2^{(j)}$, its image is zero unless $l - 2k + (h + j) = a - s, m - 2(h + j) + k = b - t$, if $-l \leq -v + h - k, k \geq h + j, v \geq 0$, then we get $v \leq a + k - s - j$. then we get $f_1^{(v)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} e_2^{(j)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) =$

$$\sum_{0 \leq p \leq k, v} v^{p(v-a-p)} \begin{bmatrix} s + p - k + j \\ p \end{bmatrix} e_2^{(h)} e_1^{(k-p)} e_2^{(j)} \xi_{(-s,-t)} \otimes f_1^{(v-p)} \eta_{(a,b)}$$

Let T denote the degree of the coefficient, then $T = p(v - a - p + s - k + j)$. If $v \leq a + k - s - j$, then $T \leq -p^2 \leq 0$, and $T = 0$ if and only if $p = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $f_1^{(v)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} e_2^{(j)}$ is fixed

by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_2^{(h)} \theta_1^{(k)} \theta_2^{(j)} \diamond \theta_1^{(v)})_{(-s,-t),(a,b)}$.

(21) For the element $e_2^{(h)} e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_1^{(v)} f_2^{(w)}$, its image is zero unless $l + 2v - w = a - s, m + 2w - v = b - t$, if $-l \geq v + k - j, -m \geq w - v + j, k \geq h + j, v \geq w$, then we get $v \geq a + w + k - s - j, w \geq b + j - t$. then we get $e_2^{(h)} e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_1^{(v)} f_2^{(w)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) = \sum_{\substack{0 \leq p \leq k, v \\ 0 \leq r \leq j \\ 0 \leq q \leq h}}$

$$v^{r(j-r-t)+p(k-p-s-j+r)+q(h-q-t-k+p+2j-2r)} \begin{bmatrix} b+r-w \\ r-d \end{bmatrix} \begin{bmatrix} b+q-w+r \\ q \end{bmatrix} \begin{bmatrix} a+p-v+w-r \\ p \end{bmatrix} e_2^{(h-q)} e_1^{(k-p)} e_2^{(j-r)} \xi_{(-s,-t)} \otimes f_1^{(v-p)} f_2^{(w-r-q)} \eta_{(a,b)}$$

Let Q denote the degree of the coefficient, then $Q = r(j-r-t) + p(k-p-s-j+r) + q(h-q-t-k+p+2j-2r) + r(b-w) + p(a-v+w-r) + q(b-w+r)$. If $v \geq a + w + k - s - j, w \geq b + j - t$, then $Q \leq -p^2 + p(a + w - v + k - s - j) + q(h + j - k) + q(-q + p - r) - r^2 + (r + q)(b - w + j - t) \leq -(q - p)^2 - qp - r^2 - qr \leq 0$

So in this case we get $Q \leq 0$ and $Q = 0$ if and only if $p = r = q = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $e_2^{(h)} e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_1^{(v)} f_2^{(w)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_2^{(h)} \theta_1^{(k)} \theta_2^{(j)} \diamond \theta_1^{(v)} \theta_2^{(w)})_{(-s,-t),(a,b)}$.

(22) For the element $f_2^{(u)} f_1^{(v)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} e_2^{(j)}$, its image is zero unless $l - 2k + (h + j) = a - s, m - 2(h + j) + k = b - t$, if $-l \leq -v + h - k, -m \leq v - u - h, k \geq h + j, v \geq u$, then we get $v \leq a + k - s - j, u \leq b + v + j + (j + h - k) - t$. then we get $f_2^{(u)} f_1^{(v)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} e_2^{(j)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) =$

$$\sum_{\substack{0 \leq p \leq k, v \\ 0 \leq q \leq u, \\ 0 \leq t' \leq q, h}} v^{p(v-a-p)+q(u-q-b-v+p)} \begin{bmatrix} s+p-k+j \\ p \end{bmatrix} \begin{bmatrix} t-h+k+q-p-2j \\ t' \end{bmatrix} \begin{bmatrix} t+q-t'-j \\ q-t' \end{bmatrix} e_2^{(h-t')} e_1^{(k-p)} e_2^{(j-q+t')} \xi_{(-s,-t)} \otimes f_2^{(u-q)} f_1^{(v-p)} \eta_{(a,b)}$$

Let S denote the degree of the coefficient, then $S = p(v - a - w - p) + q(u - q - b - v + p) + p(s - k + j) + t'(t - h + k + q - p - 2j - t') + (q - t')(t - j)$. If $v \leq a + w + k - s - j, u \leq b + v + j + (j + h - k) - t$, then $S \leq -p^2 + q(-q + p) + t'(q - p - t') + p(v - a + s + j - k) + q(j + h - k) + t'(k - j - h) \leq -p^2 +$

$$q(-q + p) + t'(q - p - t') = \begin{cases} -(q - p)^2 - qp - t'2 + (q - p)t' \leq 0 & \text{if } p \geq q \\ -p^2 - t'^2 + (t' - q)(q - p) \leq 0 & \text{if } p < q \end{cases}$$

Then we get $S \leq 0$ and $S = 0$ if and only if $p = q = t' = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $f_2^{(u)} f_1^{(v)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} e_2^{(j)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_2^{(h)} \theta_1^{(k)} \theta_2^{(j)} \diamond \theta_2^{(u)} \theta_1^{(v)})_{(-s,-t),(a,b)}$.

(23) are obviously the canonical basis elements.

(24) For the element $e_1^{(k)} 1_{(l,m)} f_2^{(v)} f_1^{(w)}$, its image is zero unless $l+2w-v = a-s, m+2v-w = b-t$, if $-l \geq k+w-v, v \geq w$, then we get $w \geq a+k-s$, then we get $e_1^{(k)} 1_{(l,m)} f_2^{(v)} f_1^{(w)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) =$

$$\sum_{0 \leq p \leq k} v^{p(k-p-s)} \begin{bmatrix} a+p-w \\ r \end{bmatrix} e_1^{(k-p)} \xi_{(-s,-t)} \otimes f_2^{(v)} f_1^{(w-p)} \eta_{(a,b)}$$

Let U denote the degree of the coefficient, then $U = p(k-p-s) + p(a-w)$. If $w \geq a+k-s$, then $H \leq -p^2 + p(a-w+k-s) \leq 0$ and $U = 0$ if and only if $p = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $e_1^{(k)} 1_{(l,m)} f_2^{(v)} f_1^{(w)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_1^{(k)} \diamond \theta_2^{(v)} \theta_1^{(w)})_{(-s,-t),(a,b)}$.

(25) For the element $f_1^{(u)} f_2^{(v)} 1_{(l,m)} e_1^{(k)}$, its image is zero unless $l-2k = a-s, m+k = b-t$, if $-l \leq v-u-k, k \geq 0, v \geq u$, then we get $u \leq a+v+k-s$.

$$\text{Then we get } f_1^{(u)} f_2^{(v)} 1_{(l,m)} e_1^{(k)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) = \sum_{0 \leq q \leq u} v^{q(u-q-a-v)} \begin{bmatrix} s+q-k \\ q \end{bmatrix} e_1^{(k-q)} \xi_{(-s,-t)} \otimes f_1^{(u-q)} f_2^{(v)} \eta_{(a,b)}$$

Let \mathbb{A} denote the degree of the coefficient, then $\mathbb{A} = q(u-q-a-v) + q(s-k)$. If $u \leq a+v+k-s$, then $\mathbb{A} = q(u-a-v+s-k) - q^2 \leq 0$, and $\mathbb{A} = 0$ if and only if $q = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $f_1^{(u)} f_2^{(v)} 1_{(l,m)} e_1^{(k)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_1^{(k)} \diamond \theta_1^{(u)} \theta_2^{(v)})_{(-s,-t),(a,b)}$.

(26) For the element $e_2^{(h)} e_1^{(k)} 1_{(l,m)} f_2^{(v)}$, its image is zero unless $l-v = a-s, m+2v = b-t$, if $-m \geq v+h-k, h \leq k, v \geq 0$, then we get $v \geq b+h-k-t$, then we get $e_2^{(h)} e_1^{(k)} 1_{(l,m)} f_2^{(v)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) =$

$$\sum_{0 \leq r \leq h} v^{r(h-r-t-k)} \begin{bmatrix} b+r-v \\ r \end{bmatrix} e_2^{(h-r)} e_1^{(k-p)} \xi_{(-s,-t)} \otimes f_2^{(v-r)} \eta_{(a,b)}$$

Let W denote the degree of the coefficient, then $W = r(h-r-t-k) + r(b-v)$. If $v \geq b+h-k-t$, then $W = r(h-k+b-v-r-t) \leq -r^2 \leq 0$

So in this case we get $W \leq 0$ and $W = 0$ if and only if $r = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $e_2^{(h)} e_1^{(k)} 1_{(l,m)} f_2^{(v)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_2^{(h)} \theta_1^{(k)} \diamond \theta_2^{(v)})_{(-s,-t),(a,b)}$.

(27) For the element $f_2^{(v)} 1_{(l,m)} e_1^{(k)} e_2^{(j)}$, its image is zero unless $l-2k+j = a-s, m-2j+k = b-t$, if $-m \leq -v+k-j, k \geq j, v \geq 0$, then we get $v \leq b+j-t$. Then we get $f_2^{(v)} 1_{(l,m)} e_1^{(k)} e_2^{(j)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) = \sum_{0 \leq p \leq v} v^{p(v-b-p)}$

$$\begin{bmatrix} t+p-j \\ p \end{bmatrix} e_1^{(k)} e_2^{(j-p)} \xi_{(-s,-t)} \otimes f_2^{(v-p)} \eta_{(a,b)}$$

Let \mathbb{B} denote the degree of the coefficient, then $\mathbb{B} = p(v-b-p) + p(t-j)$. If $v \leq b+j-t$, then $Z = -p^2 + p(v-b+t-j) \leq 0$, and $\mathbb{B} = 0$ if and only if $p = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $f_2^{(v)} 1_{(l,m)} e_1^{(k)} e_2^{(j)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_1^{(k)} \theta_2^{(j)} \diamond \theta_2^{(v)})_{(-s,-t),(a,b)}$.

(28) For the element $e_2^{(h)} e_1^{(k)} 1_{(l,m)} f_2^{(v)} f_1^{(w)}$, its image is zero unless $l+2w-v = a-s, m+2v-w = b-t$, if $-l \geq w-v+k, -m \geq v+h-k, h \leq k, v \geq w$, then we get $v \geq b+w+h-k-t, w \geq a+k-s$, then we get $e_2^{(h)} e_1^{(k)} 1_{(l,m)} f_2^{(v)} f_1^{(w)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) = \sum_{\substack{0 \leq p \leq k, w \\ 0 \leq r \leq h}} v^{p(k-p-s)+r(h-r-t-k+p)}$

$$\begin{bmatrix} a+p-w \\ p \end{bmatrix} \begin{bmatrix} b+r-v+w-p \\ r \end{bmatrix} e_2^{(h-r)} e_1^{(k-p)} \xi_{(-s,-t)} \otimes f_2^{(v-r)} f_1^{(w-p)} \eta_{(a,b)}$$

Let V denote the degree of the coefficient, then $V = p(k-p-s) + r(h-r-t-k+p) + p(a-w) + r(b-v+w-p)$. If $v \geq b+w+h-k-t, w \geq a+k-s$, then $V = p(a-w+k-s-p) + r(h-k+b-v+w-r-t) \leq -p^2 - r^2 \leq 0$

So in this case we get $V \leq 0$ and $V = 0$ if and only if $p = r = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $e_2^{(h)} e_1^{(k)} 1_{(l,m)} f_2^{(v)} f_1^{(w)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_2^{(h)} \theta_1^{(k)} \diamond \theta_2^{(v)} \theta_1^{(w)})_{(-s,-t),(a,b)}$.

(29) For the element $f_1^{(u)} f_2^{(v)} 1_{(l,m)} e_1^{(k)} e_2^{(j)}$, its image is zero unless $l-2k+j = a-s, m-2j+k = b-t$, if $-l \leq v-u-k, -m \leq -v+k-j, k \geq j, v \geq u$, then we get $v \leq b+j-t, u \leq a+v+k-s-j$. Then we get $f_1^{(u)} f_2^{(v)} 1_{(l,m)} e_1^{(k)} e_2^{(j)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) = \sum_{\substack{0 \leq p \leq v \\ 0 \leq q \leq u}} v^{p(v-b-p)+q(u-q-a-v+p)}$

$$\begin{bmatrix} s+q-k+j-p \\ q \end{bmatrix} \begin{bmatrix} t+p-j \\ p \end{bmatrix} e_1^{(k-q)} e_2^{(j-p)} \xi_{(-s,-t)} \otimes f_1^{(u-q)} f_2^{(v-p)} \eta_{(a,b)}$$

Let Z denote the degree of the coefficient, then $Z = p(v-b-p) + q(u-q-a-v+p) + p(t-j) + q(s-k+j-p)$. If $v \leq b+j-t, u \leq a+v+k-s-j$, then $Z = -p^2 + p(v-b+t-j) + q(u-a-v+s+j-k) - q^2 \leq 0$.

then we get $Z \leq 0$ and $Z = 0$ if and only if $p = q = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $f_1^{(u)} f_2^{(v)} 1_{(l,m)} e_1^{(k)} e_2^{(j)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_1^{(k)} \theta_2^{(j)} \diamond \theta_1^{(u)} \theta_2^{(v)})_{(-s,-t),(a,b)}$.

(30) For the element $e_2^{(h)} e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_2^{(v)}$, its image is zero unless $l-v = a-s, m+2v = b-t$, if $-m \geq v+j, k \geq h+j, v \geq 0$, then we get $v \geq b+j-t$. Then we get $e_2^{(h)} e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_2^{(v)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) =$

$$\sum_{\substack{0 \leq f \leq j \\ 0 \leq r \leq h}} v^{f(j-f-t)+r(h-r-t-k+2j-2f)} \begin{bmatrix} b+r-v+f \\ r \end{bmatrix} \begin{bmatrix} b+f-v \\ f \end{bmatrix} e_2^{(h-r)} e_1^{(k)} e_2^{(j-f)} \xi_{(-s,-t)} \otimes f_2^{(v-f-r)} \eta_{(a,b)}$$

Let Y denote the degree of the coefficient, then $X = f(j - f - t) + r(h - r - t - k + 2j - 2f) + f(b - v) + r(b - v + f)$. If $v \geq b + j - t$, then $X \leq -f^2 + (r + f)(b - v + j - t) + r(-r - f) + r(h + j - k) \leq -f^2 - r^2 - rf \leq 0$, and $Y = 0$ if and only if $r = f = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $e_2^{(h)} e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_2^{(v)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_2^{(h)} \theta_1^{(k)} \theta_2^{(j)} \diamond \theta_2^{(v)})_{(-s,-t),(a,b)}$.

(31) For the element $f_2^{(v)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} e_2^{(j)}$, its image is zero unless $l - 2k + (h + j) = a - s, m - 2(h + j) + k = b - t$, if $-m \leq -v - h, k \geq h + j, v \geq 0$, then we get $v \leq b + j + (j + h - k) - t$, then we get

$$f_2^{(v)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} e_2^{(j)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) = \sum_{\substack{0 \leq p \leq v \\ 0 \leq t' \leq p, h}} v^{p(v-b-p)} \begin{bmatrix} t + p - t' - j \\ p - t' \end{bmatrix} \begin{bmatrix} t - h + k - 2j + p \\ t' \end{bmatrix} e_2^{(h-t')} e_1^{(k)} e_2^{(j-p+t')} \xi_{(-s,-t)} \otimes f_2^{(v-p)} \eta_{(a,b)}$$

Let \mathbb{D} denote the degree of the coefficient, then $\mathbb{D} = p(v - b - p) + t'(t - h + k - 2j + p - t') + (p - t')(t - j)$. If $v \leq b + j + (j + h - k) - t$, then $\mathbb{D} \leq -p^2 + p(v - b + t - j) + t'(k - j - h) + t'(p - t') \leq (t' - p)p - t'^2 + (t' - p)(k - j - h)$.

Then we get $\mathbb{D} \leq 0$ and $\mathbb{D} = 0$ if and only if $p = t' = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $f_2^{(v)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} e_2^{(j)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_2^{(h)} \theta_1^{(k)} \theta_2^{(j)} \diamond \theta_2^{(v)})_{(-s,-t),(a,b)}$.

(32) For the element $e_2^{(h)} e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_2^{(v)} f_1^{(w)}$, its image is zero unless $l + 2w - v = a - s, m + 2v - w = b - t$, if $-l \geq w - v + k - j, -m \geq v + j, k \geq h + j, v \geq w$, then we get $v \geq b + w + j - t, w \geq a + k - s - j$. then we get $e_2^{(h)} e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_2^{(v)} f_1^{(w)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) =$

$$\sum_{\substack{0 \leq f \leq j \\ 0 \leq p \leq k \\ 0 \leq r \leq h}} v^{f(j-f-t)+p(k-p-s-j+f)+r(h-r-t-k+p+2j-2f)} \begin{bmatrix} b + r - v + f + w - p \\ r \end{bmatrix} \begin{bmatrix} b + f - v + w \\ f \end{bmatrix} \begin{bmatrix} a + p - w \\ p \end{bmatrix} e_2^{(h-r)} e_1^{(k-p)} e_2^{(j-f)} \xi_{(-s,-t)} \otimes f_2^{(v-f-r)} f_1^{(w-p)} \eta_{(a,b)}$$

Let X denote the degree of the coefficient, then $X = f(j - f - t) + p(k - p - s - j + f) + r(h - r - t - k + p + 2j - 2f) + f(b - v + w) + p(a - w) + r(b - v + f + w - p)$. If $v \geq b + w + j - t, w \geq a + k - s - j$, then $X \leq -f^2 + f(b - v + w + j - t) + p(-p + f) + p(a - w + k - s - j) + r(b - v + w + j - t - r - f) + r(h + j - k) \leq -(p - f)^2 - pf - r^2 - rf \leq 0$, and $X = 0$ if and only if $p = r = f = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $e_2^{(h)} e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_2^{(v)} f_1^{(w)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_2^{(h)} \theta_1^{(k)} \theta_2^{(j)} \diamond \theta_2^{(v)} \theta_1^{(w)})_{(-s,-t),(a,b)}$.

(33) For the element $f_1^{(u)} f_2^{(v)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} e_2^{(j)}$, its image is zero unless

$l-2k+(h+j) = a-s, m-2(h+j)+k = b-t$, if $-l \leq h-k+v-u, -m \leq -v-h, k \geq h+j, v \geq u$, then we get $v \leq b+j+(j+h-k)-t, u \leq a+v+k-s-j$. then we get $f_1^{(u)} f_2^{(v)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} e_2^{(j)} (\xi_{(-s,-t)} \otimes \eta_{(a,b)}) =$

$$\sum_{\substack{0 \leq q \leq u \\ 0 \leq p \leq v \\ 0 \leq t' \leq p, h}} v^{p(v-b-p)+q(u-q-a-v+p)} \begin{bmatrix} s+q-k+j-p+t' \\ q \end{bmatrix} \begin{bmatrix} t+p-t'-j \\ p-t' \end{bmatrix}$$

$$\begin{bmatrix} t-h+k-2j+p \\ t' \end{bmatrix} e_2^{(h-t')} e_1^{(k-q)} e_2^{(j-p+t')} \xi_{(-s,-t)} \otimes f_1^{(u-q)} f_2^{(v-p)} \eta_{(a,b)}$$

Let \mathbb{C} denote the degree of the coefficient, then $\mathbb{C} = p(v-b-p) + q(u-q-a-v+p) + t'(t-h+k-2j+p-t') + (p-t')(t-j) + q(s-k+j-p+t')$. If $v \leq b+j+(j+h-k)-t, u \leq a+v+k-s-j$, then $\mathbb{C} \leq -p^2 + p(v-b+t-j) + t'(k-j-h) + t'(p-t') + q(u-a-v+s-k+j) + q(-q+t') \leq (t'-p)p - (q-t')^2 - qt' + q(u-a-v+s-k+j) + (t'-p)(k-j-h)$.

Then we get $\mathbb{C} \leq 0$ and $\mathbb{C} = 0$ if and only if $p = q = t' = 0$. Meanwhile this element is fixed by the involution Ψ of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, since the element $f_1^{(u)} f_2^{(v)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} e_2^{(j)}$ is fixed by $- : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$. So using the definitions, we can see that this element is $(\theta_2^{(h)} \theta_1^{(k)} \theta_2^{(j)} \diamond \theta_1^{(u)} \theta_2^{(v)})_{(-s,-t),(a,b)}$.

3 Conjectures on some polynomial elements

We conjecture the following polynomial elements belong to the canonical basis

$$\sum_{0 \leq p \leq h} (-1)^p \begin{bmatrix} m+u+h-k+p-1 \\ p \end{bmatrix} e_2^{(h-p)} e_1^{(k)} 1_{(l-p,m+2p)} f_2^{(u-p)} f_1^{(v)} f_2^{(w)},$$

$-l \geq v+k-u, (u+h-k)+(u+w-v) \leq -m \leq u+h-k, h \leq k, v \geq u+w$.

$$\sum_{0 \leq p+q \leq h} (-1)^{p+q} \begin{bmatrix} m+u+h-k+q+p-1 \\ p \end{bmatrix} \begin{bmatrix} m+u+h-k+u+w-v+q-1 \\ q \end{bmatrix}$$

$$e_2^{(h-p-q)} e_1^{(k)} 1_{(l-p-q,m+2p+2q)} f_2^{(u-p)} f_1^{(v)} f_2^{(w-q)} + \sum_{\substack{1 \leq p \leq h \\ 0 \leq p+q \leq h}} (-1)^{p+q}$$

$$\begin{bmatrix} l+m+h+u+w+p-1 \\ p \end{bmatrix} \begin{bmatrix} m+u+h-k+u+w-v+p+q-1 \\ q \end{bmatrix}$$

$$e_2^{(h-p-q)} e_1^{(k-p)} 1_{(l+p-q,m+p+2q)} f_2^{(u)} f_1^{(v-p)} f_2^{(w-p-q)},$$

$-l \geq v+k-u, -l-m \leq u+w+h, h \leq k, v \geq u+w$.

$$\sum_{0 \leq p+q \leq h} (-1)^{p+q} \begin{bmatrix} m+u+h-k+q+p-1 \\ p \end{bmatrix} \begin{bmatrix} m+u+h-k+u+w-v+q-1 \\ q \end{bmatrix}$$

$$e_2^{(h-p-q)} e_1^{(k)} 1_{(l-p-q, m+2p+2q)} f_2^{(u-p)} f_1^{(v)} f_2^{(w-q)},$$

$$-l - m \geq u + w + h, -m \leq (u + h - k) + (u + w - v), h \leq k, v \geq u + w.$$

$$\sum_{0 \leq p \leq h} (-1)^p \begin{bmatrix} w - m + h + p - 1 \\ p \end{bmatrix} f_2^{(u)} f_1^{(v)} f_2^{(w-p)} 1_{(l+p, m-2p)} e_2^{(h-p)} e_1^{(k)},$$

$$-l \leq w - v + h - k, -w - h \leq -m \leq -w - h + v - u - w, h \leq k, v \geq u + w.$$

$$\sum_{0 \leq p+q \leq h} (-1)^{p+q} \begin{bmatrix} w + h - m + q + p - 1 \\ p \end{bmatrix} \begin{bmatrix} w + h - m + u + w - v + q - 1 \\ q \end{bmatrix} f_2^{(u-q)} f_1^{(v)} f_2^{(w-p)} 1_{(l+p+q, m-2p-2q)} e_2^{(h-p-q)} e_1^{(k)},$$

$$-l \leq w - v + h - k, -m \geq -w - h + v - u - w, h \leq k, v \geq u + w.$$

$$\sum_{0 \leq p \leq k} (-1)^p \begin{bmatrix} l + k + u + p - 1 \\ p \end{bmatrix} e_1^{(k-p)} 1_{(l+2p, m-p)} f_1^{(u-p)} f_2^{(v)} f_1^{(w)},$$

$$(u + w - v) + u + k \leq -l \leq u + k, v \geq u + w.$$

$$\sum_{0 \leq p \leq k} (-1)^p \begin{bmatrix} l + k + u + p - 1 \\ p \end{bmatrix} e_2^{(h)} e_1^{(k-p)} 1_{(l+2p, m-p)} f_1^{(u-p)} f_2^{(v)} f_1^{(w)},$$

$$(u+w-v)+u+k \leq -l \leq u+k, -m \geq v-u+h-k, v \geq u+w, h \leq k, v \geq u+w.$$

$$\sum_{0 \leq p+q \leq k} (-1)^{p+q} \begin{bmatrix} l + k + u + q + p - 1 \\ p \end{bmatrix} \begin{bmatrix} l + k + u + u + w - v + q - 1 \\ q \end{bmatrix}$$

$$e_2^{(h)} e_1^{(k-p-q)} 1_{(l+2p+2q, m-p-q)} f_1^{(u-p)} f_2^{(v)} f_1^{(w-q)},$$

$$-l \leq (u + w - v) + u + k, -m \geq v - u + h - k, v \geq u + w, h \leq k, v \geq u + w.$$

$$\sum_{0 \leq p \leq k} (-1)^p \begin{bmatrix} w - l + k - h + p - 1 \\ p \end{bmatrix} f_1^{(u)} f_2^{(v)} f_1^{(w-p)} 1_{(l-2p, m+p)} e_2^{(h)} e_1^{(k-p)},$$

$$-w + h - k \leq -l \leq -w + h - k + v - u - w, -m \leq w - v - h, h \leq k, v \geq u + w.$$

$$\begin{aligned}
& \sum_{0 \leq p+q \leq k} (-1)^{p+q} \begin{bmatrix} w-l+k-h+q+p-1 \\ p \end{bmatrix} \begin{bmatrix} w-l+k-h+u+w-v+q-1 \\ q \end{bmatrix} \\
& f_1^{(u-q)} f_2^{(v)} f_1^{(w-p)} 1_{(l-2p-2q, m+p+q)} e_2^{(h)} e_1^{(k-p-q)} + \sum_{\substack{1 \leq p \leq h \\ 0 \leq p+q \leq k}} (-1)^{p+q} \\
& \begin{bmatrix} u+w-l-m+k+p-1 \\ p \end{bmatrix} \begin{bmatrix} w-l+k-h+u+w-v+p+q-1 \\ q \end{bmatrix} \\
& f_1^{(u-p-q)} f_2^{(v-p)} f_1^{(w)} 1_{(l-p-2q, m-p+q)} e_2^{(h-p)} e_1^{(k-p-q)},
\end{aligned}$$

$$-l-m \geq -u-w-k, -m \leq w-v-h, h \leq k, v \geq u+w.$$

$$\begin{aligned}
& \sum_{0 \leq p+q \leq k} (-1)^{p+q} \begin{bmatrix} w-l+k-h+q+p-1 \\ p \end{bmatrix} \begin{bmatrix} w-l+k-h+u+w-v+q-1 \\ q \end{bmatrix} \\
& f_1^{(u-q)} f_2^{(v)} f_1^{(w-p)} 1_{(l-2p-2q, m+p+q)} e_2^{(h)} e_1^{(k-p-q)}
\end{aligned}$$

$$-l-m \leq -u-w-k, -l \geq -w+h-k+v-u-w, h \leq k, v \geq u+w.$$

$$\sum_{0 \leq p \leq j} (-1)^p \begin{bmatrix} w-m+j-k+p-1 \\ p \end{bmatrix} f_2^{(u)} f_1^{(v)} f_2^{(w-p)} 1_{(l+p, m-2p)} e_1^{(k)} e_2^{(j-p)},$$

$$-l \leq w-v-k, -w+k-h \leq -m \leq -w+k-h+(v-u-w), j \leq k, v \geq u+w.$$

$$\begin{aligned}
& \sum_{0 \leq p+q \leq j} (-1)^{p+q} \begin{bmatrix} -m+w+j-k+q+p-1 \\ p \end{bmatrix} \begin{bmatrix} -m+w+j-k+u+w-v+q-1 \\ q \end{bmatrix} \\
& f_2^{(u-q)} f_1^{(v)} f_2^{(w-p)} 1_{(l+p+q, m-2p-2q)} e_1^{(k)} e_2^{(j-p-q)} + \sum_{\substack{1 \leq p \leq j \\ 0 \leq p+q \leq j}} (-1)^{p+q} \\
& \begin{bmatrix} -l-m+j+u+w+p-1 \\ p \end{bmatrix} \begin{bmatrix} -m+w+j-k+u+w-v+p+q-1 \\ q \end{bmatrix} \\
& f_2^{(u-p-q)} f_1^{(v-p)} f_2^{(w)} 1_{(l-p+q, m-p-2q)} e_1^{(k-p)} e_2^{(j-p-q)},
\end{aligned}$$

$$-l \leq w-v-k, -l-m \geq -u-w-j, j \leq k, v \geq u+w.$$

$$\sum_{0 \leq p+q \leq j} (-1)^{p+q} \begin{bmatrix} -m+w+j-k+q+p-1 \\ p \end{bmatrix} \begin{bmatrix} -m+w+j-k+u+w-v+q-1 \\ q \end{bmatrix}$$

$$f_2^{(u-q)} f_1^{(v)} f_2^{(w-p)} 1_{(l+p+q, m-2p-2q)} e_1^{(k)} e_2^{(j-p-q)},$$

$$-l - m \leq -u - w - j, -m \geq (k - j - w) + (v - u - w), j \leq k, v \geq u + w.$$

$$\sum_{0 \leq p \leq j} (-1)^p \begin{bmatrix} m + u + p - 1 \\ p \end{bmatrix} e_1^{(k)} e_2^{(j-p)} 1_{(l-p, m+2p)} f_2^{(u-p)} f_1^{(v)} f_2^{(w)},$$

$$-l \geq v - u + k - j, u + j + (u + w - v) \leq -m \leq u + j, j \leq k, v \geq u + w.$$

$$\sum_{0 \leq p+q \leq j} (-1)^{p+q} \begin{bmatrix} u + j + m + q + p - 1 \\ p \end{bmatrix} \begin{bmatrix} u + j + m + u + w - v + q - 1 \\ q \end{bmatrix} e_1^{(k)} e_2^{(j-p-q)} 1_{(l-p-q, m+2p+2q)} f_2^{(u-p)} f_1^{(v)} f_2^{(w-q)},$$

$$-l \geq v - u + k - j, -m \leq u + j + u + w - v, j \leq k, v \geq u + w.$$

$$\sum_{0 \leq p \leq k} (-1)^p \begin{bmatrix} -l + k + w + p - 1 \\ p \end{bmatrix} f_1^{(u)} f_2^{(v)} f_1^{(w-p)} 1_{(l-2p, m+p)} e_1^{(k-p)},$$

$$-w - k \leq -l \leq -w - k + (v - u - w), v \geq u + w.$$

$$\sum_{0 \leq p \leq k} (-1)^p \begin{bmatrix} -l + k + w + p - 1 \\ p \end{bmatrix} f_1^{(u)} f_2^{(v)} f_1^{(w-p)} 1_{(l-2p, m+p)} e_1^{(k-p)} e_2^{(j)},$$

$$-w - k \leq -l \leq -w - k + (v - u - w), -m \leq w - v + k - j, v \geq u + w, j \leq k, v \geq u + w.$$

$$\sum_{0 \leq p+q \leq k} (-1)^{p+q} \begin{bmatrix} -l + k + w + q + p - 1 \\ p \end{bmatrix} \begin{bmatrix} -l + k + w + u + w - v + q - 1 \\ q \end{bmatrix}$$

$$f_1^{(u-q)} f_2^{(v)} f_1^{(w-p)} 1_{(l-2p-2q, m+p+q)} e_1^{(k-p-q)} e_2^{(j)},$$

$$-l \geq -w - k + (v - u - w), -m \leq w - v + k - j, v \geq u + w, j \leq k, v \geq u + w.$$

$$\sum_{0 \leq p \leq k} (-1)^p \begin{bmatrix} u + l + k - j + p - 1 \\ p \end{bmatrix} e_1^{(k-p)} e_2^{(j)} 1_{(l-2p, m+p)} f_1^{(u-p)} f_2^{(v)} f_1^{(w)},$$

$$u + k - j + (u + w - v) \leq -l \leq u + k - j, -m \geq v - u + j, j \leq k, v \geq u + w.$$

$$\begin{aligned}
& \sum_{0 \leq p+q \leq k} (-1)^{p+q} \begin{bmatrix} u+l+k-j+q+p-1 \\ p \end{bmatrix} \begin{bmatrix} u+l+k-j+u+w-v+q-1 \\ q \end{bmatrix} \\
& e_1^{(k-p-q)} e_2^{(j)} 1_{(l+2p+2q, m-p-q)} f_1^{(u-p)} f_2^{(v)} f_1^{(w-q)} + \sum_{\substack{1 \leq p \leq j \\ 0 \leq p+q \leq k}} (-1)^{p+q} \\
& \begin{bmatrix} u+w+l+m+k+p-1 \\ p \end{bmatrix} \begin{bmatrix} u+l+k-j+u+w-v+p+q-1 \\ q \end{bmatrix} \\
& e_1^{(k-p-q)} e_2^{(j-p)} 1_{(l+p+2q, m+p-q)} f_1^{(u)} f_2^{(v-p)} f_1^{(w-p-q)}, \\
& -l-m \leq u+w+k, -m \geq v-u+j, j \leq k, v \geq u+w.
\end{aligned}$$

$$\begin{aligned}
& \sum_{0 \leq p+q \leq k} (-1)^{p+q} \begin{bmatrix} u+l+k-j+q+p-1 \\ p \end{bmatrix} \begin{bmatrix} u+l+k-j+u+w-v+q-1 \\ q \end{bmatrix} \\
& e_1^{(k-p-q)} e_2^{(j)} 1_{(l+2p+2q, m-p-q)} f_1^{(u-p)} f_2^{(v)} f_1^{(w-q)} \\
& -l \leq u+k-j+(u+w-v), -l-m \geq u+w+k, j \leq k, v \geq u+w.
\end{aligned}$$

Remark The key of the proof is to write the canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ corresponding to these polynomial elements. For example, if we can prove the following identity:

$$\begin{aligned}
& \sum_{0 \leq p \leq h} (-1)^p \begin{bmatrix} m+u+h-k+p-1 \\ p \end{bmatrix} e_2^{(h-p)} e_1^{(k)} 1_{(l-p, m+2p)} f_2^{(u-p)} f_1^{(v)} f_2^{(w)} (\xi_{(-s, -t)} \otimes \\
& \eta_{(a, b)}) = \sum_{\substack{0 \leq p \leq k, v \\ 0 \leq q \leq h \\ 0 \leq t' \leq q, u}} v^{p(k-p-s)+t'(u+2w-b-v+p-q)+(q-t')(h-k+p-q-t)} \begin{bmatrix} a+p-v+w \\ p \end{bmatrix} \\
& \begin{bmatrix} k-h+t+q-p \\ t' \end{bmatrix} \begin{bmatrix} b+q-t'-w \\ q-t' \end{bmatrix} e_2^{(h-q)} e_1^{(k-p)} \xi_{(-s, -t)} \otimes f_2^{(u-t')} f_1^{(v-p)} f_2^{(w-q+t')} \eta_{(a, b)}
\end{aligned}$$

if $-l \geq v+k-u, (u+h-k)+(u+w-v) \leq -m \leq u+h-k, h \leq k, v \geq u+w$, then we get $v \geq a+w+k-s, u+2w \leq b+v+h-k-t, w \geq b+h-k-t$, we can check that the degree \mathbb{A} of the coefficient of the right hand of the identity above satisfy $\mathbb{A} \leq 0$ and $\mathbb{A} = 0$ if and only if $p = q = t' = 0$.

However in general it is very tedious to write the canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ corresponding to these polynomial elements.

4 More conjectures

As can be seen above, the author computed the canonical basis elements of $\bar{\mathbf{U}}$ by definition. The most important procedure to work out it is that you must write the canonical basis of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$, from

which you can get the canonical basis of $\dot{\mathbf{U}}$ for type A_2 . According to the author's computation, when you write $(\theta_2^{(h)}\theta_1^{(k)}\theta_2^{(j)}\diamond\theta_2^{(u)}\theta_1^{(v)}\theta_2^{(w)})_{(-s,-t),(a,b)}$. The coefficients \mathbb{A} is divided into the following five domains

$$\begin{cases} v \geq a + w + k - s - j & \begin{cases} u + 2w \geq b + v + j + (j + h - k) - t \\ u + 2w \leq b + v + j + (j + h - k) - t \end{cases} \\ v \leq a + w + k - s - j & \begin{cases} u + 2w \geq b + v + j - t \\ u + 2w \leq b + v + j - t \end{cases} \end{cases} \begin{cases} w \geq b + j + (j + h - k) - t & \begin{cases} w \geq b + j - t \\ w \leq b + j - t \end{cases} \\ w \leq b + j + (j + h - k) - t & \begin{cases} w \geq b + j - t \\ w \leq b + j - t \end{cases} \end{cases}$$

When you take $v \geq a + w + k - s - j, u + 2w \geq b + v + j + (j + h - k) - t, u + 2w \geq b + v + j - t, w \geq b + j + (j + h - k) - t, w \geq b + j - t$, i.e. $v \geq a + w + k - s - j, u + 2w \geq b + v + j - t$, then these canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ correspond to the monomial elements $e_2^{(h)}e_1^{(k)}e_2^{(j)}1_{(l,m)}f_2^{(u)}f_1^{(v)}f_2^{(w)}(-l \geq v + k - j - u, -m \geq u + j, k \geq h + j, v \geq u + w.)$ of $\dot{\mathbf{U}}$.

When you take $v \leq a + w + k - s - j, u + 2w \leq b + v + j + (j + h - k) - t, u + 2w \leq b + v + j - t, w \leq b + j + (j + h - k) - t, w \leq b + j - t$, i.e. $v \leq a + w + k - s - j, w \leq b + j + (j + h - k) - t$, then these canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ correspond to the monomial elements $f_2^{(u)}f_1^{(v)}f_2^{(w)}1_{(l,m)}e_2^{(h)}e_1^{(k)}e_2^{(j)}(-l \leq w - v + h - k, -m \leq -w - h, k \geq h + j, v \geq u + w.)$ of $\dot{\mathbf{U}}$.

The other cases just correspond to the polynomial elements of the canonical basis which can be seen as follows.

If $v \geq a + w + k - s - j, u + 2w \geq b + v + j + (j + h - k) - t, w \geq b + j - t, u + 2w \leq b + v + j - t$, then these canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ correspond to the polynomial elements of $\dot{\mathbf{U}}$:

$$\sum_{0 \leq p \leq j} (-1)^p \begin{bmatrix} m + u + j + p - 1 \\ p \end{bmatrix} e_2^{(h)}e_1^{(k)}e_2^{(j-p)}1_{(l-p,m+2p)}f_2^{(u-p)}f_1^{(v)}f_2^{(w)},$$

$-l \geq v - u + k - j, u + j + (u + w - v) \leq -m \leq u + j, -m \geq u + j + (j + h - k), k \geq j + h, v \geq u + w.$

Applying to the antiautomorphism σ , then we get the polynomial elements of $\dot{\mathbf{U}}$:

$$\sum_{0 \leq p \leq h} (-1)^p \begin{bmatrix} w - m + h + p - 1 \\ p \end{bmatrix} f_2^{(u)}f_1^{(v)}f_2^{(w-p)}1_{(l+p,m-2p)}e_2^{(h-p)}e_1^{(k)}e_2^{(j)},$$

$-l \leq w - v + h - k, -w - h \leq -m \leq -w - h + v - u - w, -m \leq -w - h + (k - j - h), k \geq j + h, v \geq u + w.$

Which correspond to these canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ when $v \leq a + w + k - s - j, u + 2w \leq b + v + j + (j + h - k) - t, b + j + (j + h - k) - t \leq w \leq b + j - t.$

If $v \geq a + w + k - s - j, u + 2w \leq b + v + j + (j + h - k) - t, w \geq b + j - t, u + 2w \leq b + v + j - t$, then these canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ correspond to the polynomial elements of $\dot{\mathbf{U}}$:

$$\sum_{\substack{0 \leq p \leq j \\ 0 \leq q \leq h}} (-1)^{p+q} \begin{bmatrix} m + u + j + q + p - 1 \\ p \end{bmatrix} \begin{bmatrix} m + u + j + (j + h - k) + q - 1 \\ q \end{bmatrix} \\ e_2^{(h-q)} e_1^{(k)} e_2^{(j-p)} 1_{(l-p-q, m+2p+2q)} f_2^{(u-p-q)} f_1^{(v)} f_2^{(w)}, \\ -l \geq v - u + k - j, -m \geq u + j + (u + w - v), -m \leq u + j + (j + h - k), k \geq j + h, v \geq u + w.$$

Applying to the antiautomorphism σ , then we get the polynomial elements of $\dot{\mathbf{U}}$:

$$\sum_{\substack{0 \leq p \leq h \\ 0 \leq q \leq j}} (-1)^{p+q} \begin{bmatrix} w - m + h + q + p - 1 \\ p \end{bmatrix} \begin{bmatrix} w - m + h + (j + h - k) + q - 1 \\ p \end{bmatrix} \\ f_2^{(u)} f_1^{(v)} f_2^{(w-p-q)} 1_{(l+p+q, m-2p-2q)} e_2^{(h-p)} e_1^{(k)} e_2^{(j-q)}, \\ -l \leq w - v + h - k, -m \leq -w - h + v - u - w, -m \geq -w - h + (k - j - h), k \geq j + h, v \geq u + w.$$

Which correspond to these canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ when $v \leq a + w + k - s - j, u + 2w \leq b + v + j + (j + h - k) - t, w \geq b + j - t$.

If $v \geq a + w + k - s - j, u + 2w \geq b + v + j + (j + h - k) - t, w \leq b + j - t, u + 2w \leq b + v + j - t$, then these canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ correspond to the polynomial elements of $\dot{\mathbf{U}}$:

$$\sum_{0 \leq p+q \leq j} (-1)^{p+q} \begin{bmatrix} u + j + m + q + p - 1 \\ p \end{bmatrix} \begin{bmatrix} u + j + m + u + w - v + q - 1 \\ q \end{bmatrix} \\ e_2^{(h)} e_1^{(k)} e_2^{(j-p-q)} 1_{(l-p-q, m+2p+2q)} f_2^{(u-p)} f_1^{(v)} f_2^{(w-q)}, \\ -l \geq v - u + k - j, -m \leq u + j + u + w - v, -m \geq u + j + (j + h - k), k \geq j + h, v \geq u + w.$$

Applying to the antiautomorphism σ , then we get the polynomial elements of $\dot{\mathbf{U}}$:

$$\sum_{0 \leq p+q \leq h} (-1)^{p+q} \begin{bmatrix} w + h - m + q + p - 1 \\ p \end{bmatrix} \begin{bmatrix} w + h - m + u + w - v + q - 1 \\ q \end{bmatrix} \\ f_2^{(u-q)} f_1^{(v)} f_2^{(w-p)} 1_{(l+p+q, m-2p-2q)} e_2^{(h-p-q)} e_1^{(k)} e_2^{(j)}, \\ -l \leq w - v + h - k, -m \geq -w - h + v - u - w, -m \leq -w - h + (k - j - h), k \geq j + h, v \geq u + w.$$

Which correspond to these canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ when $v \leq a + w + k - s - j, u + 2w \geq b + v + j + (j + h - k) - t, w \leq b + j - t$.

If $v \geq a + w + k - s - j, u + 2w \leq b + v + j + (j + h - k) - t, w \leq b + j - t, w \geq b + j + (j + h - k) - t$, then these canonical basis elements of

$V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ correspond to the polynomial elements of $\dot{\mathbf{U}}$:

$$\sum_{\substack{0 \leq p+q \leq j \\ 0 \leq r \leq h}} (-1)^{p+q+r} \begin{bmatrix} u+j+m+r+q+p-1 \\ p \end{bmatrix} \begin{bmatrix} u+j+m+u+w-v+q-1 \\ q \end{bmatrix} \begin{bmatrix} m+u+j+(j+h-k)+r-1 \\ r \end{bmatrix} e_2^{(h-r)} e_1^{(k)} e_2^{(j-p-q)} 1_{(l-p-q-r, m+2p+2q+2r)} f_2^{(u-p-r)} f_1^{(v)} f_2^{(w-q)},$$

$$-l \geq v-u+k-j, -m \leq u+j+u+w-v, u+j+(j+h-k)+(u+w-v) \leq$$

$$-m \leq u+j+(j+h-k), k \geq j+h, v \geq u+w.$$

Applying to the antiautomorphism σ , then we get the polynomial elements of $\dot{\mathbf{U}}$:

$$\sum_{\substack{0 \leq p+q \leq h \\ 0 \leq r \leq j}} (-1)^{p+q+r} \begin{bmatrix} w+h-m+r+q+p-1 \\ p \end{bmatrix} \begin{bmatrix} w+h-m+u+w-v+q-1 \\ q \end{bmatrix} \begin{bmatrix} w-m+h+(j+h-k)+r-1 \\ r \end{bmatrix} f_2^{(u-q)} f_1^{(v)} f_2^{(w-p-r)} 1_{(l+p+q+r, m-2p-2q-2r)} e_2^{(h-p-q)} e_1^{(k)} e_2^{(j-r)},$$

$$-l \leq w-v+h-k, -m \geq -w-h+v-u-w, -w-h+(k-j-h) \leq$$

$$-m \leq -w-h+(k-j-h)+(v-u-w), k \geq j+h, v \geq u+w.$$

Which correspond to these canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ when $v \leq a+w+k-s-j, u+2w \geq b+v+j+(j+h-k)-t, w \geq b+j-t, u+2w \leq b+v+j-t$.

The most complicated case is that if you change four signs, i.e. if $v \geq a+w+k-s-j, w \leq b+v+j+(j+h-k)-t, w \leq b+j-t$.

If $v \geq a+w+k-s-j, w \leq b+v+j+(j+h-k)-t, v \leq a+b+j+h-s-t$, then these canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ correspond to the polynomial elements of $\dot{\mathbf{U}}$:

$$\sum_{\substack{0 \leq p+q \leq j \\ 0 \leq r+i \leq h}} (-1)^{p+q+r+i} \begin{bmatrix} u+j+m+r+i+q+p-1 \\ p \end{bmatrix} \begin{bmatrix} u+j+m+u+w-v+q-1 \\ q \end{bmatrix} \begin{bmatrix} m+u+j+(j+h-k)+i+r-1 \\ r \end{bmatrix} \begin{bmatrix} m+u+j+(j+h-k)+(u+w-v)+i-1 \\ i \end{bmatrix}$$

$$e_2^{(h-r-i)} e_1^{(k)} e_2^{(j-p-q)} 1_{(l-p-q-r-i, m+2p+2q+2r+2i)} f_2^{(u-p-r)} f_1^{(v)} f_2^{(w-q-i)} +$$

$$\sum_{\substack{0 \leq p+q \leq j \\ 1 \leq r \leq h \\ 0 \leq i \leq h-r}} (-1)^{p+q+r+i} \begin{bmatrix} u+j+m+r+i+q+p-1 \\ p \end{bmatrix} \begin{bmatrix} u+j+m+u+w-v+q-1 \\ q \end{bmatrix} \begin{bmatrix} l+m+j+h+u+w+r-1 \\ r \end{bmatrix} \begin{bmatrix} m+u+j+(j+h-k)+(u+w-v)+i+r-1 \\ i \end{bmatrix}$$

$$e_2^{(h-r-i)} e_1^{(k-r)} e_2^{(j-p-q)} 1_{(l-p-q-i+r, m+2p+2q+r+2i)} f_2^{(u-p)} f_1^{(v-r)} f_2^{(w-q-r-i)},$$

$$-l \geq v-u+k-j, -l-m \leq j+h+u+w, k \geq j+h, v \geq u+w.$$

Applying to the antiautomorphism σ , then we get the polynomial elements of $\dot{\mathbf{U}}$:

$$\sum_{\substack{0 \leq p+q \leq h \\ 0 \leq r+i \leq j}} (-1)^{p+q+r+i} \begin{bmatrix} w+h-m+r+i+q+p-1 \\ p \end{bmatrix} \begin{bmatrix} w+h-m+u+w-v+q-1 \\ q \end{bmatrix}$$

$$\begin{aligned}
& \left[\begin{array}{c} w - m + h + (j + h - k) + i + r - 1 \\ r \end{array} \right] \left[\begin{array}{c} w - m + h + (j + h - k) + (u + w - v) + i - 1 \\ i \end{array} \right] \\
& f_2^{(u-q-i)} f_1^{(v)} f_2^{(w-p-r)} 1_{(l+p+q+r+i, m-2p-2q-2r-2i)} e_2^{(h-p-q)} e_1^{(k)} e_2^{(j-r-i)} + \\
& \sum_{\substack{0 \leq p+q \leq h \\ 1 \leq r \leq j \\ 0 \leq i \leq j-r}} (-1)^{p+q+r+i} \left[\begin{array}{c} w + h - m + r + i + q + p - 1 \\ p \end{array} \right] \left[\begin{array}{c} w + h - m + u + w - v + q - 1 \\ q \end{array} \right] \\
& \left[\begin{array}{c} -l - m + j + h + u + w + r - 1 \\ r \end{array} \right] \left[\begin{array}{c} w - m + h + (j + h - k) + (u + w - v) + i + r - 1 \\ i \end{array} \right] \\
& f_2^{(u-q-r-i)} f_1^{(v-r)} f_2^{(w-p)} 1_{(l+p+q+i-r, m-2p-2q-r-2i)} e_2^{(h-p-q)} e_1^{(k-r)} e_2^{(j-r-i)}, \\
& -l \leq w - v + h - k, -l - m \geq -j - h - u - w, k \geq j + h, v \geq u + w.
\end{aligned}$$

Which correspond to these canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ when $v \leq a + w + k - s - j, u + w \geq a + b + k - s - t, u + 2w \geq b + v + j - t$.

If $v \geq a + w + k - s - j, w \leq b + v + j + (j + h - k) - t, v \geq a + b + j + h - s - t$, then these canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ correspond to the polynomial elements of $\dot{\mathbf{U}}$:

$$\begin{aligned}
& \sum_{\substack{0 \leq p+q \leq j \\ 0 \leq r+i \leq h}} (-1)^{p+q+r+i} \left[\begin{array}{c} u + j + m + r + i + q + p - 1 \\ p \end{array} \right] \left[\begin{array}{c} u + j + m + u + w - v + q - 1 \\ q \end{array} \right] \\
& \left[\begin{array}{c} m + u + j + (j + h - k) + i + r - 1 \\ r \end{array} \right] \left[\begin{array}{c} m + u + j + (j + h - k) + (u + w - v) + i - 1 \\ i \end{array} \right] \\
& e_2^{(h-r-i)} e_1^{(k)} e_2^{(j-p-q)} 1_{(l-p-q-r-i, m+2p+2q+2r+2i)} f_2^{(u-p-r)} f_1^{(v)} f_2^{(w-q-i)}, \\
& -m \leq u + j + (j + h - k + (u + w - v)), -l - m \geq j + h + u + w, k \geq j + h, v \geq u + w.
\end{aligned}$$

Applying to the antiautomorphism σ , then we get the polynomial elements of $\dot{\mathbf{U}}$:

$$\begin{aligned}
& \sum_{\substack{0 \leq p+q \leq h \\ 0 \leq r+i \leq j}} (-1)^{p+q+r+i} \left[\begin{array}{c} w + h - m + r + i + q + p - 1 \\ p \end{array} \right] \left[\begin{array}{c} w + h - m + u + w - v + q - 1 \\ q \end{array} \right] \\
& \left[\begin{array}{c} w - m + h + (j + h - k) + i + r - 1 \\ r \end{array} \right] \left[\begin{array}{c} w - m + h + (j + h - k) + (u + w - v) + i - 1 \\ i \end{array} \right] \\
& f_2^{(u-q-i)} f_1^{(v)} f_2^{(w-p-r)} 1_{(l+p+q+r+i, m-2p-2q-2r-2i)} e_2^{(h-p-q)} e_1^{(k)} e_2^{(j-r-i)}, \\
& -m \geq -w - h + (k - j - h) + (v - u - w), -l - m \leq -j - h - u - w, k \geq j + h, v \geq u + w.
\end{aligned}$$

Which correspond to these canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ when $v \leq a + w + k - s - j, u + w \leq a + b + k - s - t, u + 2w \geq b + v + j - t$.

When you write $(\theta_2^{(h)} \theta_1^{(k)} \theta_2^{(j)} \diamond \theta_1^{(u)} \theta_2^{(v)} \theta_1^{(w)})_{(-s, -t), (a, b)}$. The coefficients \mathbb{B} is divided into the following four domains

$$\begin{cases} v \geq b + w + j - t & \begin{cases} v \geq b + w + j + (j + h - k) - t \\ v \leq b + w + j + (j + h - k) - t \end{cases} \\ v \leq b + w + j - t & \begin{cases} v \geq b + w + j + (j + h - k) - t \\ v \leq b + w + j + (j + h - k) - t \end{cases} \end{cases}$$

$$\begin{cases} u+2w \geq a+v+k-s-j \\ u+2w \geq a+v+k-s-j \end{cases} \quad \begin{cases} w \geq a+k-s-j \\ w \leq a+k-s-j \end{cases}$$

When you take $v \geq b+w+j-t, v \geq b+w+j+(j+h-k)-t, u+2w \geq a+v+k-s-j, w \geq a+k-s-j$ i.e. $v \geq b+w+j-t, u+2w \geq a+v+k-s-j$, then these canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ correspond to the monomial elements $e_2^{(h)} e_1^{(k)} e_2^{(j)} 1_{(l,m)} f_1^{(u)} f_2^{(v)} f_1^{(w)} (-l \geq u+k-j, -m \geq j+v-u, k \geq h+j, v \geq u+w.)$ of $\dot{\mathbf{U}}$.

When you take $v \leq b+w+j-t, v \leq b+w+j+(j+h-k)-t, u+2w \leq a+v+k-s-j, w \leq a+k-s-j$, i.e. $v \leq b+w+j+(j+h-k)-t, w \leq a+k-s-j$, then these canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ correspond to the monomial elements $f_1^{(u)} f_2^{(v)} f_1^{(w)} 1_{(l,m)} e_2^{(h)} e_1^{(k)} e_2^{(j)} (-l \leq h-k-w, -m \leq w-v-h, k \geq h+j, v \geq u+w.)$ of $\dot{\mathbf{U}}$.

The other cases just correspond to the polynomial elements of the canonical basis which can be seen as follows.

If $b+w+j+(j+h-k)-t \leq v \leq b+w+j-t, u+2w \geq a+v+k-s-j$, then these canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ correspond to the polynomial elements of $\dot{\mathbf{U}}$:

$$\sum_{0 \leq p \leq j} (-1)^p \begin{bmatrix} m+j+v-u+p-1 \\ p \end{bmatrix} e_2^{(h)} e_1^{(k)} e_2^{(j-p)} 1_{(l-p, m+2p)} f_1^{(u)} f_2^{(v-p)} f_1^{(w)},$$

$-l \geq u+k-j, v+j-u+(j+h-k) \leq -m \leq v+j-u, k \geq j+h, v \geq u+w.$

Applying to the antiautomorphism σ , then we get the polynomial elements of $\dot{\mathbf{U}}$:

$$\sum_{0 \leq p \leq h} (-1)^p \begin{bmatrix} -m+h+v-w+p-1 \\ p \end{bmatrix} f_1^{(u)} f_2^{(v-p)} f_1^{(w)} 1_{(l+p, m-2p)} e_2^{(h-p)} e_1^{(k)} e_2^{(j)},$$

$-l \leq -w+h-k, -h+w-v \leq -m \leq -h+w-v+(k-j-h), k \geq j+h, v \geq u+w.$

Which correspond to these canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ when $b+w+j+(j+h-k)-t \leq v \leq b+w+j-t, w \leq a+k-s-j$.

If $v \geq b+w+j-t, u+2w \leq a+v+k-s-j, w \geq a+k-s-j$, then these canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ correspond to the polynomial elements of $\dot{\mathbf{U}}$:

$$\sum_{0 \leq p \leq k} (-1)^p \begin{bmatrix} l+u+k-j+p-1 \\ p \end{bmatrix} e_2^{(h)} e_1^{(k-p)} e_2^{(j)} 1_{(l+2p, m-p)} f_1^{(u-p)} f_2^{(v)} f_1^{(w)},$$

$u+k-j+(u+w-v) \leq -l \geq u+k-j, -m \geq v+j-u, k \geq j+h, v \geq u+w.$

Applying to the antiautomorphism σ , then we get the polynomial elements of $\dot{\mathbf{U}}$:

$$\sum_{0 \leq p \leq k} (-1)^p \begin{bmatrix} -l+w+k-h+p-1 \\ p \end{bmatrix} f_1^{(u)} f_2^{(v)} f_1^{(w-p)} 1_{(l-2p, m+p)} e_2^{(h)} e_1^{(k-p)} e_2^{(j)},$$

$-w+h-k \leq -l \leq -w+h-k+(v-u-w), -m \leq -h+w-v, k \geq$

$j + h, v \geq u + w$.

Which correspond to these canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ when $v \leq b + w + j + (j + h - k) - t, u + 2w \leq a + v + k - s - j, w \geq a + k - s - j$.

If $b + w + j + (j + h - k) - t \leq v \leq b + w + j - t, u + 2w \leq a + v + k - s - j, w \geq a + k - s - j$, then these canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ correspond to the polynomial elements of $\dot{\mathbf{U}}$:

$$\sum_{\substack{0 \leq p \leq j \\ 0 \leq q \leq k}} (-1)^{p+q} \begin{bmatrix} m + j + v - u + p - 1 \\ p \end{bmatrix} \begin{bmatrix} l + u + k - j + q - 1 \\ q \end{bmatrix} e_2^{(h)} e_1^{(k-q)} e_2^{(j-p)} 1_{(l-p+2q, m+2p-q)} f_1^{(u-q)} f_2^{(v-p)} f_1^{(w)},$$

$$u + k - j + (u + w - v) \leq -l \leq u + k - j, v + j - u + (j + h - k) \leq -m \leq v + j - u, k \geq j + h, v \geq u + w.$$

Applying to the antiautomorphism σ , then we get the same polynomial elements of $\dot{\mathbf{U}}$, because they correspond to these canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ when $b + w + j + (j + h - k) - t \leq v \leq b + w + j - t, u + 2w \leq a + v + k - s - j, w \geq a + k - s - j$.

The most complicated case is that if $u + 2w \geq a + v + k - s - j, v \leq b + w + j + (j + h - k) - t$,

If $u + 2w \geq a + v + k - s - j, v \leq b + w + j + (j + h - k) - t, u + w \geq a + b + j + h - s - t$, then these canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ correspond to the polynomial elements of $\dot{\mathbf{U}}$:

$$\sum_{0 \leq p+q \leq k} (-1)^{p+q} \begin{bmatrix} w - l + k - h + q + p - 1 \\ p \end{bmatrix} \begin{bmatrix} w - l + k - h + u + w - v + q - 1 \\ q \end{bmatrix} f_1^{(u-q)} f_2^{(v)} f_1^{(w-p)} 1_{(l-2p-2q, m+p+q)} e_2^{(h)} e_1^{(k-p-q)} e_2^{(j)} + \sum_{\substack{1 \leq p \leq h \\ 0 \leq p+q \leq k}} (-1)^{p+q}$$

$$\begin{bmatrix} u + w - l - m + k + p - 1 \\ p \end{bmatrix} \begin{bmatrix} w - l + k - h + u + w - v + p + q - 1 \\ q \end{bmatrix} f_1^{(u-p-q)} f_2^{(v-p)} f_1^{(w)} 1_{(l-p-2q, m-p+q)} e_2^{(h-p)} e_1^{(k-p-q)} e_2^{(j)},$$

$$-l - m \geq -u - w - k, -m \leq w - v + j - h, h + j \leq k, v \geq u + w.$$

Applying to the antiautomorphism σ , then we get the polynomial elements of $\dot{\mathbf{U}}$:

$$\sum_{0 \leq p+q \leq k} (-1)^{p+q} \begin{bmatrix} u + l + k - j + q + p - 1 \\ p \end{bmatrix} \begin{bmatrix} u + l + k - j + u + w - v + q - 1 \\ q \end{bmatrix} e_2^{(h)} e_1^{(k-p-q)} e_2^{(j)} 1_{(l+2p+2q, m-p-q)} f_1^{(u-p)} f_2^{(v)} f_1^{(w-q)} + \sum_{\substack{1 \leq p \leq j \\ 0 \leq p+q \leq k}} (-1)^{p+q}$$

$$\begin{bmatrix} u + w + l + m + k + p - 1 \\ p \end{bmatrix} \begin{bmatrix} u + l + k - j + u + w - v + p + q - 1 \\ q \end{bmatrix}$$

$$e_2^{(h)} e_1^{(k-p-q)} e_2^{(j-p)} 1_{(l+p+2q, m+p-q)} f_1^{(u)} f_2^{(v-p)} f_1^{(w-p-q)},$$

$$-l-m \leq u+w+k, -m \geq v-u+j-h, j+h \leq k, v \geq u+w.$$

Which correspond to these canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ when $v \geq b+w+j-t, v \leq a+b+k-s-t, w \leq a+k-s-j$.

If $u+2w \geq a+v+k-s-j, v \leq b+w+j+(j+h-k)-t, u+w \leq a+b+j+h-s-t$, then these canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ correspond to the polynomial elements of $\dot{\mathbf{U}}$:

$$\sum_{0 \leq p+q \leq k} (-1)^{p+q} \begin{bmatrix} w-l+k-h+q+p-1 \\ p \end{bmatrix} \begin{bmatrix} w-l+k-h+u+w-v+q-1 \\ q \end{bmatrix}$$

$$f_1^{(u-q)} f_2^{(v)} f_1^{(w-p)} 1_{(l-2p-2q, m+p+q)} e_2^{(h)} e_1^{(k-p-q)} e_2^{(j)},$$

$$-l-m \leq -u-w-k, -l \geq -w+h-k+(v-u-w), h+j \leq k, v \geq u+w.$$

Applying to the antiautomorphism σ , then we get the polynomial elements of $\dot{\mathbf{U}}$:

$$\sum_{0 \leq p+q \leq k} (-1)^{p+q} \begin{bmatrix} u+l+k-j+q+p-1 \\ p \end{bmatrix} \begin{bmatrix} u+l+k-j+u+w-v+q-1 \\ q \end{bmatrix}$$

$$e_2^{(h)} e_1^{(k-p-q)} e_2^{(j)} 1_{(l+2p+2q, m-p-q)} f_1^{(u-p)} f_2^{(v)} f_1^{(w-q)},$$

$$-l-m \geq u+w+k, -l \leq u+k-j+(u+w-v), j+h \leq k, v \geq u+w.$$

Which correspond to these canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ when $v \geq b+w+j-t, v \geq a+b+k-s-t, w \leq a+k-s-j$.

We have a similar discussion about the canonical basis elements $(\theta_1^{(h)} \theta_2^{(k)} \theta_1^{(j)} \diamond \theta_1^{(u)} \theta_2^{(v)} \theta_1^{(w)})_{(-s, -t), (a, b)}$ and $(\theta_1^{(h)} \theta_2^{(k)} \theta_1^{(j)} \diamond \theta_2^{(u)} \theta_1^{(v)} \theta_2^{(w)})_{(-s, -t), (a, b)}$. And from them we get the other canonical elements of $\dot{\mathbf{U}}$, we will not write them here.

Conjecture: All these polynomial elements are belonging to the canonical basis $\dot{\mathbf{B}}$ and complete.

Remark1: The key to prove all these polynomial elements belong to the canonical basis $\dot{\mathbf{B}}$ is to write the canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ corresponding to these polynomial elements. When changing one sign it may be easy, however in general it is very tedious to write the canonical basis elements of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$ corresponding to these polynomial elements.

Remark2: In Theorem 2, we easily see that σ gives a one to one correspondence between the elements in (1) and (2), (2) and (6), and so on. Hence completeness is probably true if we can prove that they are belonging to the canonical basis $\dot{\mathbf{B}}$.

5 The quasi-R-matrix

The quasi- R -matrix Θ was introduced in [L3,L4,theorem 4.1.2], which plays an important role to write the canonical basis of $V(-s\omega_1 - t\omega_2) \otimes V(a\omega_1 + b\omega_2)$. For type A_1 , Lusztig had given in [L3]. For type A_2 , the author has calculated a part of summations of Θ :

$$(-1)^t [s-t]! [t]! (v-v^{-1})^s v^{-(s-t)(s-t-3)/2-t(t-1)/2} f_1^{(s)} f_2^{(s-t)} \otimes \left(\sum_{t'=0}^{s-t} (-1)^{t'} (v^{-t-1})^{t'} e_2^{(t')} e_1^{(s)} e_2^{(s-t-t')} \right).$$

of course we also have:

$$(-1)^t [s-t]! [t]! (v-v^{-1})^s v^{-(s-t)(s-t-3)/2-t(t-1)/2} f_2^{(s)} f_1^{(s-t)} \otimes \left(\sum_{t'=0}^{s-t} (-1)^{t'} (v^{-t-1})^{t'} e_1^{(t')} e_2^{(s)} e_1^{(s-t-t')} \right).$$

$$\text{and } (-1)^t [s-t]! [t]! (v-v^{-1})^s v^{-(s-t)(s-t-3)/2-t(t-1)/2} f_2^{(s-t)} f_1^{(s)} \otimes \left(\sum_{t'=0}^{s-t} (-1)^{t'} (v^{-t-1})^{t'} e_2^{(s-t-t')} e_1^{(s)} e_2^{(t')} \right).$$

$$(-1)^t [s-t]! [t]! (v-v^{-1})^s v^{-(s-t)(s-t-3)/2-t(t-1)/2} f_1^{(s-t)} f_2^{(s)} \otimes \left(\sum_{t'=0}^{s-t} (-1)^{t'} (v^{-t-1})^{t'} e_1^{(s-t-t')} e_2^{(s)} e_1^{(t')} \right).$$

The author has also calculated some more: $(-1)^{s+1} [s-2]! [2] v^{-(s-1)(s-2)/2} (v-v^{-1})^s f_2 f_1^{(s)} f_2 \otimes (e_1^{(s)} e_2^{(2)} - \frac{[s+1]-[s-1]}{[2]} e_2 e_1^{(s)} e_2 + e_2^{(2)} e_1^{(s)})$. and $(-1)^{s+1} [s-2]! [2] v^{-(s-1)(s-2)/2} (v-v^{-1})^s f_1 f_2^{(s)} f_1 \otimes (e_2^{(s)} e_1^{(2)} - \frac{[s+1]-[s-1]}{[2]} e_1 e_2^{(s)} e_1 + e_1^{(2)} e_2^{(s)})$.
 $-[3]! v^{-1} (v-v^{-1})^3 f_2^{(2)} f_1^{(3)} f_2 \otimes (e_2^{(3)} e_1^{(3)} - v \cdot \frac{v^2+v^{-4}+v^{-2}}{[3]} e_2^{(2)} e_1^{(3)} e_2 + v \cdot \frac{v+v^{-1}+v^{-5}}{[3]} e_2 e_1^{(3)} e_2^{(2)} - v^{-1} e_1^{(3)} e_2^{(3)})$.

$$(-1)^t [s-t+1]! [t-1]! v^{-t(t-1)/2-(s-t)(s-t-1)/2} (v-v^{-1})^s f_2^{(s-t)} f_1^{(s)} f_2 \otimes (-1)^{t'} v^{s-t} \frac{v^{t-s-t(t'-1)} [s+1]-v^{t+1-(t+2)t'} [t]}{[s-t+1]} e_2^{(s-t+1-t')} e_1^{(s)} e_2^{(t')} \text{ for } t \geq 1, s \geq t+1.$$

$$[4]! v^{-4} (v-v^{-1})^4 f_2^{(2)} f_1^{(4)} f_2^{(2)} \otimes (e_2^{(4)} e_1^{(4)} - v^4 \cdot \frac{[2](v^{-1}+v^{-7})}{[4]} e_2^{(3)} e_1^{(4)} e_2 + v^4 \cdot \frac{[2](v^2+v^{-2}+2v^{-4}+v^{-6}+v^{-10})}{[3][4]} e_2^{(2)} e_1^{(4)} e_2^{(2)} - v^4 \cdot \frac{[2](v^{-1}+v^{-7})}{[4]} e_2 e_1^{(4)} e_2^{(3)} + e_1^{(4)} e_2^{(4)}).$$

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